THE STOKES OPERATOR WITH NEUMANN BOUNDARY CONDITIONS IN LIPSCHITZ DOMAINS

MARIUS MITREA, SYLVIE MONNIAUX AND MATTHEW WRIGHT

ABSTRACT. In the first part of the paper we give a satisfactory definition of the Stokes operator in Lipschitz domains in $\mathbb{R}^n$ when boundary conditions of Neumann type are considered. We then proceed to establish optimal global Sobolev regularity results for vector fields in the domains of fractional powers of this Neumann-Stokes operator. Finally, we study existence, regularity and uniqueness of mild solutions of the Navier-Stokes system with Neumann boundary conditions.

CONTENTS

1. INTRODUCTION

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$, and fix a finite number $T > 0$. The Navier-Stokes equations are the standard system of PDE's governing the flow of continuum matter in fluid form, such as liquid or gas, occupying the domain Ω. These equations describe the change with respect to time $t \in [0, T]$ of the velocity and pressure of the fluid. A widely used version of the Navier-Stokes initial boundary problem, equipped with a Dirichlet

Supported in part by NSF and a UMC Miller Scholar Award
2000 *Mathematics Subject Classification.* Primary: 35Q30, 76D07, Secondary: 35A15, 35Q10, 76D03
Key words and phrases. Stokes operator, Neumann boundary conditions, Lipschitz domains, domain of fractional power, regularity, Sobolev spaces, Navier-Stokes equations.

boundary condition, reads

$$
\left\{
\begin{array}{rcll}
\dfrac{\partial \vec{u}}{\partial t} - \Delta_x \vec{u} + \nabla_x \pi + (\vec{u}\cdot\nabla_x)\vec{u} & = & 0 & \text{in} \quad (0,T]\times\Omega, \\
\mathrm{div}_x\, \vec{u} & = & 0 & \text{in} \quad [0,T]\times\Omega, \\
\mathrm{Tr}_x\, \vec{u} & = & 0 & \text{on} \quad [0,T]\times\partial\Omega, \\
\vec{u}(0) & = & \vec{u}_0 & \text{in} \quad \Omega,
\end{array}
\right. \tag{1.1}
$$

where $\vec{u}$ is the velocity field and π denotes the pressure of the fluid. One of the strategies for dealing with (1.1), brought to prominence by the pioneering work of H. Fujita, and T. Kato in the 60's, consists of recasting (1.1) in the form of an abstract initial value problem

$$
\left\{
\begin{array}{ll}
\vec{u}'(t) + (A\vec{u})(t) = \vec{f}(t) & t\in(0,T), \\
\vec{f}(t) := -\mathbb{P}_D\big[(\vec{u}(t)\cdot\nabla_x)\vec{u}(t)\big], & \\
\vec{u}(0) = \vec{u}_0, &
\end{array}
\right. \tag{1.2}
$$

which is then converted into the integral equation

$$
\vec{u}(t) = e^{-tA}\vec{u}_0 - \int_0^t e^{-(t-s)A}\mathbb{P}_D\Big[(\vec{u}(s)\cdot\nabla_x)\vec{u}(s)\Big]\,ds, \qquad 0<t<T, \tag{1.3}
$$

then finally solving (1.3) via fixed point methods (typically, a Picard iterative scheme). In this scenario, the operator $\mathbb{P}_D$ is the Leray (orthogonal) projection of $L^2(\Omega)^n$ onto the space $\mathcal{H}_D := \{\vec{u}\in L^2(\Omega)^n : \mathrm{div}\,\vec{u}=0 \text{ in } \Omega,\ \nu\cdot\vec{u}=0 \text{ on } \partial\Omega\}$, where ν is the outward unit normal to Ω, and A is the Stokes operator, i.e. the Friedrichs extension of the symmetric operator $\mathbb{P}_D\circ(-\Delta_D)$, where Δ_D is the Dirichlet Laplacian, to an unbounded self-adjoint operator on the space $\mathcal{H}_D$.

By relying on the theory of analytic semigroups generated by self-adjoint operators, Fujita and Kato have proved in [11] short time existence of strong solutions for (1.1) when $\Omega\subset\mathbb{R}^3$ is bounded and sufficiently smooth. Somewhat more specifically, they have shown that if Ω is a bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega$ of class $\mathscr{C}^3$, and if the initial datum $\vec{u}_0$ belongs to $D(A^{\frac14})$, then a strong solution can be found for which $\vec{u}(t)\in D(A^{\frac34})$ for $t\in(0,T)$, granted that T is small. Hereafter, $D(A^\alpha)$, $\alpha>0$, stands for the domain of the fractional power A^α of A.

An important aspect of this analysis is the ability to describe the size/smoothness of vector fields belonging to $D(A^\alpha)$ in terms of more familiar spaces. For example, the estimates (1.18) and (2.17) in [11] amount to

$$
D(A^\gamma)\subset\mathscr{C}^\alpha(\overline{\Omega})^3 \quad \text{if} \ \ \tfrac34<\gamma<1 \ \text{ and } \ 0<\alpha<2(\gamma-\tfrac34), \tag{1.4}
$$

which plays a key role in [11]. Although Fujita and Kato have proved (1.4) via *ad hoc* methods, it was later realized that a more resourceful and elegant approach to such regularity results is to view them as corollaries of optimal embeddings for $D(A^\alpha)$, $\alpha>0$, into the scale of vector-valued Sobolev (potential) spaces of fractional order, $L^p_s(\Omega)^3$, $1<p<\infty$, $s\in\mathbb{R}$. This latter issue turned out to be intimately linked to the smoothness assumptions made on the boundary of the domain Ω. For example, Fujita and Morimoto have proved in [12] that

$$
\partial\Omega\in\mathscr{C}^\infty \Longrightarrow D(A^\alpha)\subset L^2_{2\alpha}(\Omega)^3, \qquad 0\le\alpha\le1, \tag{1.5}
$$

whereas the presence of a single conical singularity on $\partial\Omega$ may result in the failure of $D(A)$ to be included in $L^2_2(\Omega)^3$.

The issue of extending the Fujita-Kato approach to the class of Lipschitz domains has been recently resolved in [29]. In the process, several useful global Sobolev regularity results for the vector fields in the fractional powers of the Stokes operator have been established. For example, it has been proved in [29] that for any Lipschitz domain Ω in $\mathbb{R}^3$,

$$D(A^{\frac{3}{4}}) \subset L^p_{\frac{3}{p}}(\Omega)^3 \quad \forall\, p > 2, \tag{1.6}$$

$$\forall\, \alpha > \tfrac{3}{4} \quad \exists\, p > 3 \ \text{ such that } \ D(A^{\alpha}) \subset L^p_1(\Omega)^3, \tag{1.7}$$

$$D(A^{\gamma}) = L^2_{2\gamma,z}(\Omega)^3 \cap \mathcal{H}_D, \qquad 0 < \gamma < \tfrac{3}{4}, \tag{1.8}$$

where, if $s > 0$, $L^2_{s,z}(\Omega)$ is the subspace of $L^2_s(\Omega)$ consisting of functions whose extension by zero outside Ω belongs to $L^2_s(\mathbb{R}^n)$. Also, it was shown in [29] that for any Lipschitz domain Ω in $\mathbb{R}^3$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that

$$\tfrac{3}{4} < \gamma < \tfrac{3}{4} + \varepsilon \Longrightarrow D(A^{\gamma}) \subset \mathscr{C}^{2\gamma - 3/2}(\overline{\Omega})^3, \tag{1.9}$$

in agreement with the Fujita-Kato regularity result (1.4). For related work, as well as further pertinent references, the reader is referred to, e.g., O.A. Ladyzhenskaya [22], R. Temam [41], M.E. Taylor [39], Y. Giga and T. Miyakawa [13], and W. von Wahl [43].

The aim of this paper is to derive analogous results in the case when Neumann-type boundary conditions are considered in place of the Dirichlet boundary condition. Dictated by specific practical considerations (such as phenomena translating into free boundary problems), several scenarios are possible. For example, the Neumann condition

$$(\nabla \vec{u} + \nabla \vec{u}^{\top})\nu - \pi\nu = 0 \ \text{ on } \ (0,T) \times \partial\Omega, \tag{1.10}$$

(recall that ν stands for the outward unit normal to $\partial\Omega$) has been frequently used in the literature. From a physical point of view, it is convenient to view (1.10) as $T(\vec{u},\pi)\nu = 0$ on $(0,T)\times\partial\Omega$, where $T(\vec{u},\pi) := \nabla\vec{u} + (\nabla\vec{u})^{\top} - \pi$ denotes the stress tensor. In other words, in the case of a free boundary, (1.10) is expressing the absence of stress on the interface separating the two media. A more detailed account in this regard can be found in, e.g., D.D. Joseph's monograph [17]. See also the articles [38], [14], [15], and the references therein. Another Neumann-type condition of interest is

$$\partial_\nu \vec{u} - \pi\nu = 0 \ \text{ on } \ (0,T) \times \partial\Omega. \tag{1.11}$$

This has been employed in [8] (in the stationary case). Here we shall work with a one-parameter family of Neumann-type boundary conditions,

$$[(\nabla\vec{u})^{\top} + \lambda\,(\nabla\vec{u})]\nu - \pi\nu = 0 \ \text{ on } \ (0,T) \times \partial\Omega, \tag{1.12}$$

indexed by $\lambda \in (-1,1]$ (in this context, (1.10), (1.11) correspond to choosing $\lambda = 1$ and $\lambda = 0$, respectively). Much as in the case of the Fujita-Kato approach for (1.1), a basic ingredient in the treatment of the initial Navier-Stokes boundary problem with Neumann boundary conditions, i.e.,

$$\left\{\begin{array}{rcll} \dfrac{\partial\vec{u}}{\partial t} - \Delta_x\vec{u} + \nabla_x\pi + (\vec{u}\cdot\nabla_x)\vec{u} & = & 0 & \text{in} \quad (0,T]\times\Omega, \\ \operatorname{div}_x\vec{u} & = & 0 & \text{in} \quad [0,T]\times\Omega, \\ [(\nabla_x\vec{u})^{\top} + \lambda\,(\nabla_x\vec{u})]\nu - \pi\nu & = & 0 & \text{on} \quad [0,T]\times\partial\Omega, \\ \vec{u}(0) & = & \vec{u}_0 & \text{in} \quad \Omega, \end{array}\right. \tag{1.13}$$

is a suitable analogue of the Stokes operator $A = \mathbb{P}_D \circ (-\Delta_D)$ discussed earlier. As a definition for this, we propose taking the unbounded operator

$$B_\lambda : D(B_\lambda) \subset \mathcal{H}_N \longrightarrow \mathcal{H}_N, \tag{1.14}$$

where we have set $\mathcal{H}_N := \{\vec{u} \in L^2(\Omega)^n : \operatorname{div} \vec{u} = 0 \text{ in } \Omega\}$, with domain

$$\begin{aligned} D(B_\lambda) \;:=\; \Big\{ & \vec{u} \in L^2_1(\Omega)^n \cap \mathcal{H}_N : \text{ there exists } \pi \in L^2(\Omega) \text{ so that } -\Delta\vec{u} + \nabla\pi \in \mathcal{H}_N \\ & \text{and such that } [(\nabla\vec{u})^\top + \lambda\,(\nabla\vec{u})]\nu - \pi\nu = 0 \text{ on } \partial\Omega \Big\}, \end{aligned} \tag{1.15}$$

(with a suitable interpretation of the boundary condition) and acting according to

$$B_\lambda \vec{u} := -\Delta\vec{u} + \nabla\pi, \qquad \vec{u} \in D(B_\lambda), \tag{1.16}$$

In order to be able to differentiate this from the much more commonly used Stokes operator $A = \mathbb{P}_D \circ (-\Delta_D)$, we shall call the latter the *Dirichlet-Stokes operator* and refer to (1.15)-(1.16) as the *Neumann-Stokes operator*.

Let us now comment on the suitability of the Neumann-Stokes operator B_λ vis-a-vis to the solvability of the initial Navier-Stokes system with Neumann boundary conditions (1.13). To this end, denote by $\mathbb{P}_N$ the orthogonal projection of $L^2(\Omega)^n$ onto the space $\mathcal{H}_N = \{\vec{u} \in L^2(\Omega)^n : \operatorname{div}\vec{u} = 0 \text{ in } \Omega\}$. In particular,

$$\mathbb{P}_N(\nabla q) = 0 \quad \text{ for every } q \in L^2_1(\Omega) \text{ with } \operatorname{Tr} q = 0 \text{ on } \partial\Omega. \tag{1.17}$$

Proceed formally and assume that $\vec{u}$, π solve (1.13) and that q solves the inhomogeneous Dirichlet problem

$$\begin{cases} \Delta q = \Delta\pi \text{ in } \Omega, \\ q\Big|_{\partial\Omega} = 0. \end{cases} \tag{1.18}$$

Then $\nabla\pi - \nabla q$ is divergence-free. Based on this and (1.17) we may then compute

$$\mathbb{P}_N(\nabla\pi) = \mathbb{P}_N(\nabla\pi - \nabla q) = \nabla\pi - \nabla q = \nabla(\pi - q). \tag{1.19}$$

Since $\pi - q$ has the same boundary trace as π, it follows that $[(\nabla\vec{u})^\top + \lambda\,(\nabla\vec{u})]\nu - (\pi - q)\nu = 0$ on $\partial\Omega$. Consequently,

$$B_\lambda(\vec{u}) = -\Delta\,\vec{u} + \nabla(\pi - q) = \mathbb{P}_N(-\Delta\,\vec{u} + \nabla\pi). \tag{1.20}$$

Thus, when $\mathbb{P}_N$ is formally applied to the first line in (1.13) we arrive at the abstract evolution problem

$$\begin{cases} \vec{u}'(t) + (B_\lambda\vec{u})(t) = \vec{f}(t) \qquad t \in (0,T), \\ \vec{f}(t) := -\mathbb{P}_N\big[(\vec{u}(t)\cdot\nabla_x)\vec{u}(t)\big], \\ \vec{u}(0) = \vec{u}_0, \end{cases} \tag{1.21}$$

which is the natural analogue of (1.2) in the current setting. This opens the door for solving (1.13) by considering the integral equation

$$\vec{u}(t) = e^{-tB_\lambda}\vec{u}_0 - \int_0^t e^{-(t-s)B_\lambda}\mathbb{P}_N\Big[(\vec{u}(s)\cdot\nabla_x)\vec{u}(s)\Big]\,ds, \qquad 0 < t < T. \tag{1.22}$$

In fact, in analogy with [40] where a similar issue is raised for the Dirichlet-Stokes operator, we make the following:

Conjecture. *For a given bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ there exists $\varepsilon = \varepsilon(\Omega) > 0$ such that the Neumann-Stokes operator associated generates an analytic semigroup on $\{\vec{u} \in L^p(\Omega)^3 : \operatorname{div} \vec{u} = 0\}$ provided $3/2 - \varepsilon < p < 3 + \varepsilon$.*

The range of p's in the above conjecture is naturally dictated by the mapping properties of the Neumann-Leray projection which happens to extend to a well-defined bounded operator

$$\mathbb{P}_N : L^p(\Omega)^3 \longrightarrow \{u \in L^p(\Omega)^3 : \operatorname{div} u = 0\} \tag{1.23}$$

precisely for $3/2 - \varepsilon < p < 3 + \varepsilon$ where $\varepsilon = \varepsilon(\Omega) > 0$. Indeed, in a recent paper, [30], M. Mitrea and S. Monniaux have proved the version of the above conjecture corresponding to the Stokes system equipped with boundary conditions which, in the case of $\mathscr{C}^2$ domains, coincides with the standard Navier's slip boundary conditions

$$\left\{ \begin{array}{rcl} \nu \cdot \vec{u} & = & 0 \quad \text{on} \quad (0,T) \times \partial\Omega \\ {[(\nabla \vec{u} + \nabla \vec{u}^\top)\nu]_{tan}} & = & 0 \quad \text{on} \quad (0,T) \times \partial\Omega, \end{array} \right. \tag{1.24}$$

if one neglects surface tension effects (responsible for a zero-order term, involving the curvature of the boundary).

In summary, the interest in the functional analytic properties of the Neumann-Stokes operator B_λ in (1.15)-(1.16) is justified.

We establish sharp global Sobolev regularity results for vector fields in $D(B_\lambda^\alpha)$, the domain of fractional powers of B_λ. Our main results in this regard parallel those for the Dirichlet-Stokes operator which have been reviewed in the first part of the introduction. For the sake of this introduction, we wish to single out several such results. Concretely, for a Lipschitz domain Ω in $\mathbb{R}^n$ we show that

$$D(B_\lambda^{\frac{s}{2}}) = \Big\{ \vec{u} \in L^2_s(\Omega)^n : \operatorname{div} \vec{u} = 0 \ \text{ in } \ \Omega \Big\} \ \text{ if } \ 0 \le s \le 1, \tag{1.25}$$

and

$$D(B_\lambda^\alpha) \subset \bigcup_{p > \frac{2n}{n-1}} L^p_1(\Omega)^n \quad \text{if} \quad \alpha > \tfrac{3}{4}. \tag{1.26}$$

Also, when $n = 3$,

$$D(B_\lambda^\alpha) \subset \mathscr{C}^{2\alpha - 3/2}(\bar{\Omega})^3 \quad \text{if} \quad \tfrac{3}{4} < \alpha < \tfrac{3}{4} + \varepsilon, \tag{1.27}$$

$$D(B_\lambda^{\frac{3}{4}}) \subset L^3_1(\Omega)^3, \tag{1.28}$$

and when $n = 2$,

$$D(B_\lambda^\alpha) \subset \mathscr{C}^{2\alpha - 1}(\bar{\Omega})^2 \quad \text{if} \quad \tfrac{3}{4} < \alpha < \tfrac{3}{4} + \varepsilon, \tag{1.29}$$

for some small $\varepsilon = \varepsilon(\Omega) > 0$.

It should be noted that, in the case when $\partial\Omega \in \mathscr{C}^\infty$, the initial boundary value problem (1.13) has been treated (when $\lambda = 1$) by G. Grubb and V. Solonnikov in [14], [15], [38] (cf. also the references therein for relevant, earlier work). In this scenario, the typical departure point is the regularity result $D(B_1) \subset L^2_2(\Omega)^n$, which nonetheless is irreconcilably false in the class of Lipschitz domains considered here. Most importantly, the pseudo-differential methods used in these references are no longer applicable in the non-smooth setting we treat. We wish to emphasize that overcoming the novel, significant difficulties caused by allowing domains with irregular boundaries represents the main technical achievement of the current paper.

Key ingredients in the proof of the regularity results (1.25)-(1.28) are the sharp results for the well-posedness of the inhomogeneous problem for the Stokes operator equipped with Neumann boundary conditions in a Lipschitz domain Ω in $\mathbb{R}^n$, with data from Besov and Triebel-Lizorkin spaces from [32]. This yields a clear picture of the nature of $D(B_\lambda)$. On the other hand, known abstract functional analytic results allow us to identify $D(B_\lambda^{1/2})$. Starting from these, other intermediate fractional powers can then be treated by relying on certain (non-standard) interpolation techniques.

The organization of the paper is as follows. In Section 2 we collect a number of preliminary results of function theoretic nature. Section 3 is devoted to a discussion of the meaning and properties of the conormal derivative $[(\nabla\vec{u})^\top + \lambda\,(\nabla\vec{u})]\nu - \pi\nu$ on $\partial\Omega$ when $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and $\vec{u}$, π belong to certain Besov-Triebel-Lizorkin spaces. Section 4 is reserved for a review of the definitions and properties of linear operators associated with sesquilinear forms. Next, in Section 5, we collect some basic abstract results about semigroups and fractional powers of self-adjoint operators. The rigorous definition of the Neumann-Stokes operator B_λ is given in Section 6. Among other things, here we show that B_λ is self-adjoint on $\mathcal{H}_N$ and identify $D(B_\lambda^{1/2})$. The scale $V^{p,s}(\Omega) := \{\vec{u} \in L^p_s(\Omega)^n : \operatorname{div}\vec{u} = 0\}$ is investigated in Section 7 where we show that, for certain ranges of indices, this is stable under complex interpolation and duality. In Section 8 we record an optimal, well-posedness result for the Poisson problem for the Stokes system with Neumann-type boundary conditions in Lipschitz domains, with data from Besov-Triebel-Lizorkin spaces, recently established in [32]. The global Sobolev regularity of vector fields belonging to $D(B_\lambda^\alpha)$ for $\alpha \in [0,1]$ is discussed in Section 9 and Section 10, when the underlying domain is Lipschitz. Finally, in the last section, we treat the solvability of (1.22), thus complementing results obtained in [29] for the Stokes operator equipped with Dirichlet boundary conditions.

Acknowledgments. This work has been completed while the authors had been visiting Université Aix-Marseille 3 and the University of Missouri-Columbia, whose hospitality they wish to gratefully acknowledge.

2. Preliminaries

We shall call an open, bounded, nonempty set, with connected boundary $\Omega \subset \mathbb{R}^n$ a *Lipschitz domain* if for every point $x^* \in \partial\Omega$ there is a rotation of the Euclidean coordinates in $\mathbb{R}^n$, a neighborhood $\mathcal{O}$ of x^* and a Lipschitz function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap \mathcal{O} = \{x = (x', x_n) \in \mathbb{R}^n : x_n > \varphi(x')\} \cap \mathcal{O}. \tag{2.1}$$

In this scenario, we let $d\sigma$ stand for the surface measure on $\partial\Omega$, and denote by ν the outward unit normal to $\partial\Omega$. Next, for $k \in \mathbb{N}$ and $p \in (1,\infty)$, we recall the classical Sobolev space

$$L^p_k(\Omega) := \Big\{ f \in L^p(\Omega) : \|f\|_{W^{k,p}(\Omega)} := \sum_{|\gamma|\le k} \|\partial^\gamma f\|_{L^p(\Omega)} < \infty \Big\}, \tag{2.2}$$

(throughout the paper, all derivatives are taken in the sense of distributions) and set

$$L^p_{k,z}(\Omega) \ := \ \text{the closure of } \mathscr{C}^\infty_c(\Omega) \text{ in } L^p_k(\Omega). \tag{2.3}$$

Then for every $k \in \mathbb{N}$ and $1 < p, p' < \infty$ with $1/p + 1/p' = 1$, we have

$$L^p_{-k}(\Omega) := \Big\{ \sum_{|\gamma|\le k} \partial^\gamma f_\gamma : f_\gamma \in L^p(\Omega) \Big\} = \Big(L^{p'}_{k,z}(\Omega) \Big)^*. \tag{2.4}$$

Moving on, for $s \in (0,1)$, $1 \leq p \leq \infty$, denote by

$$B_s^{p,p}(\partial\Omega) := \Big\{ f \in L^p(\partial\Omega) : \Big(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n-1+sp}} \, d\sigma_x d\sigma_y \Big)^{\frac{1}{p}} < +\infty \Big\}, \tag{2.5}$$

the Besov class on $\partial\Omega$. We equip this with the natural norm

$$\|f\|_{B_s^{p,p}(\partial\Omega)} := \|f\|_{L^p(\partial\Omega)} + \Big(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{d-1+sp}} \, d\sigma_x d\sigma_y \Big)^{1/p}. \tag{2.6}$$

For $s \in (0,1)$ and $1 < p, p' < \infty$ with $1/p + 1/p' = 1$, we also set

$$B_{-s}^{p,p}(\partial\Omega) := \Big(B_s^{p',p'}(\partial\Omega) \Big)^*. \tag{2.7}$$

In the sequel, we shall occasionally write $L_s^2(\partial\Omega)$ in place of $B_s^{2,2}(\partial\Omega)$ for $s \in (-1,1)$.

Recall that if $p \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, then the trace operator

$$\mathrm{Tr} : L_1^p(\Omega) \longrightarrow B_{1-1/p}^{p,p}(\partial\Omega) \tag{2.8}$$

is well-defined, linear and bounded (cf. [16]).

Next, introduce

$$\mathcal{H} := \{ \vec{u} \in L^2(\Omega)^n : \operatorname{div} \vec{u} = 0 \text{ in } \Omega \} \tag{2.9}$$

which is a closed subspace of $L^2(\Omega)^n$ (hence, a Hilbert space when equipped with the norm inherited from $L^2(\Omega)^n$). Also, set

$$\mathcal{V} := L_1^2(\Omega)^n \cap \mathcal{H} \tag{2.10}$$

which is a closed subspace of $L_1^2(\Omega)^n$ hence, a reflexive Banach space when equipped with the norm inherited from $L_1^2(\Omega)^n$.

Lemma 2.1. *If $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain then*

$$\mathcal{V} \hookrightarrow \mathcal{H} \quad \textit{continuously and densely.} \tag{2.11}$$

Proof. The continuity of the inclusion mapping in (2.11) is obvious. To prove that this has a dense range, fix $\vec{u} \in \mathcal{H}$. Then it has been proved in [21] that there exists a smooth domain $\mathcal{O}$ and $\vec{w} \in L^2(\mathcal{O})^n$ with the following properties:

$$\overline{\Omega} \subset \mathcal{O}, \qquad \operatorname{div} \vec{w} = 0 \text{ in } \mathcal{O}, \qquad \vec{w}\big|_{\Omega} = u. \tag{2.12}$$

In analogy with (2.9), (2.10), set

$$\mathcal{H}(\mathcal{O}) := \{ \vec{v} \in L^2(\mathcal{O})^n : \operatorname{div} \vec{v} = 0 \text{ in } \mathcal{O} \}, \qquad \mathcal{V}(\mathcal{O}) := L_1^2(\mathcal{O})^n \cap \mathcal{H}(\mathcal{O}). \tag{2.13}$$

Then the following Hodge-Helmholtz-Weyl decompositions are valid

$$L_1^2(\mathcal{O})^n = \mathcal{V}(\mathcal{O}) \oplus \nabla \Big[L_2^2(\mathcal{O}) \cap L_{1,z}^2(\mathcal{O}) \Big], \tag{2.14}$$

$$L^2(\mathcal{O})^n = \mathcal{H}(\mathcal{O}) \oplus \Big[\nabla L_{1,z}^2(\mathcal{O}) \Big]. \tag{2.15}$$

These can be obtained constructively as follows. Granted that $\mathcal{O}$ is a smooth domain (here, it suffices to have $\partial\mathcal{O} \in \mathscr{C}^{1,r}$ for some $r > 1/2$), the Poisson problem with homogeneous Dirichlet boundary condition

$$\begin{cases} \Delta q = f \in L^2(\mathcal{O}), \\ q \in L_2^2(\mathcal{O}) \cap L_{1,z}^2(\mathcal{O}), \end{cases} \tag{2.16}$$

is well-posed, and we denote by

$$G : L^2(\mathcal{O}) \longrightarrow L_2^2(\mathcal{O}) \cap L_{1,z}^2(\mathcal{O}), \qquad Gf = q, \tag{2.17}$$

the solution operator associated with (2.16). By the Lax-Milgram lemma, the latter further extends to a bounded, self-adjoint operator

$$G : L^2_{-1}(\mathcal{O}) \longrightarrow L^2_{1,z}(\mathcal{O}). \tag{2.18}$$

With I denoting the identity operator, if we now consider

$$P := I - \nabla \circ G \circ \operatorname{div}, \tag{2.19}$$

then in each instance below

$$P : L^2_1(\mathcal{O})^n \longrightarrow \mathcal{V}(\mathcal{O}), \qquad P : L^2(\mathcal{O})^n \longrightarrow \mathcal{H}(\mathcal{O}), \tag{2.20}$$

P is a well-defined, linear and bounded operator. Furthermore, in the second case in (2.20), P actually acts as the orthogonal projection. Indeed, this is readily verified using the fact that

$$P = P^* \text{ in } L^2(\mathcal{O})^n \quad \text{and} \quad P\Big|_{\mathcal{H}(\mathcal{O})} = I, \text{ the identity operator.} \tag{2.21}$$

The Hodge-Helmholtz-Weyl decompositions (2.14)-(2.15) are then naturally induced by decomposing the identity operator according to

$$I = P + \nabla \circ G \circ \operatorname{div}, \tag{2.22}$$

both on $L^2_1(\mathcal{O})^n$ and on $L^2(\mathcal{O})^n$.

After this preamble, we now turn to the task of establishing (2.11). Choose a sequence $\vec{w}_j \in L^2_1(\mathcal{O})^n$, $j \in \mathbb{N}$, such that $\vec{w}_j \to \vec{w}$ in $L^2(\mathcal{O})^n$ as $j \to \infty$. Then $\vec{w} = P\vec{w} = \lim_{j\to\infty} P\vec{w}_j$ in $L^2(\mathcal{O})^n$ and $\vec{u}_j := [P\vec{w}_j]|_\Omega \in \mathcal{V}$ for every $j \in \mathbb{N}$. Since these considerations imply that $\vec{u} = \vec{w}|_\Omega = \lim_{j\to\infty} \vec{u}_j$ in $L^2(\Omega)^n$, (2.11) follows. $\Box$

Remark 2.2. An inspection of the above proof shows that, via a similar argument, we have that

$$P : \mathscr{C}^\infty(\overline{\Omega}) \hookrightarrow \mathcal{H} \cap \mathscr{C}^\infty(\overline{\Omega}) \quad \text{boundedly.} \tag{2.23}$$

Thus, ultimately,

$$\{\vec{u} \in \mathscr{C}^\infty(\overline{\Omega})^n : \operatorname{div} \vec{u} = 0 \text{ in } \Omega\} \hookrightarrow \mathcal{H} \quad \text{densely.} \tag{2.24}$$

Next, recall that ν stands for the outward unit normal to Ω, and introduce the following closed subspace of $L^2_{1/2}(\partial\Omega)^n$:

$$L^2_{1/2,\nu}(\partial\Omega) := \Big\{\vec{\varphi} \in L^2_{1/2}(\partial\Omega)^n : \int_{\partial\Omega} \nu \cdot \vec{\varphi}\, d\sigma = 0\Big\}. \tag{2.25}$$

Our goal is to show that the trace operator from (2.8) extends to a bounded mapping

$$\operatorname{Tr} : \mathcal{V} \longrightarrow L^2_{1/2,\nu}(\partial\Omega) \tag{2.26}$$

which is onto. In fact, it is useful to prove the following more general result.

Lemma 2.3. *Assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, with outward unit normal ν and surface measure $d\sigma$. Also, fix $1 < p < \infty$ and $s \in (1/p, 1 + 1/p)$. Then the trace operator from* (2.8) *extends to a bounded mapping*

$$\operatorname{Tr} : \Big\{\vec{u} \in L^p_s(\Omega)^n : \operatorname{div} \vec{u} = 0\Big\} \longrightarrow \Big\{\vec{\varphi} \in B^{p,p}_{s-1/p}(\partial\Omega)^n : \int_{\partial\Omega} \nu \cdot \vec{\varphi}\, d\sigma = 0\Big\}, \tag{2.27}$$

which is onto.

Proof. The fact that (2.27) is well-defined, linear and bounded is clear from the properties of (2.8) and the fact that

$$\int_{\partial\Omega} \nu \cdot \mathrm{Tr}\,\vec{u}\,d\sigma = \int_{\Omega} \mathrm{div}\,\vec{u}\,dx = 0, \tag{2.28}$$

whenever $\vec{u} \in L^p_s(\Omega)^n$ is divergence-free. To see that (2.27) is also onto, consider $\vec{\varphi} \in B^{p,p}_{s-1/p}(\partial\Omega)^n$ satisfying

$$\int_{\partial\Omega} \nu \cdot \vec{\varphi}\,d\sigma = 0 \tag{2.29}$$

and solve the divergence equation

$$\begin{cases} \mathrm{div}\,\vec{u} = 0 \text{ in } \Omega, \\ \vec{u} \in L^p_s(\Omega)^n, \\ \mathrm{Tr}\,\vec{u} = \vec{\varphi} \text{ on } \partial\Omega. \end{cases} \tag{2.30}$$

For a proof of the fact that this is solvable for any $\vec{\varphi} \in B^{p,p}_{s-1/p}(\partial\Omega)^n$ satisfying (2.29) see [27]. This shows that the operator (2.27) is indeed onto. □

Moving on, for $\lambda \in \mathbb{R}$ fixed, let

$$a^{\alpha\beta}_{jk}(\lambda) := \delta_{jk}\delta_{\alpha\beta} + \lambda\,\delta_{j\beta}\delta_{k\alpha}, \qquad 1 \le j,k,\alpha,\beta \le n, \tag{2.31}$$

and, adopting the summation convention over repeated indices, consider the differential operator L_λ given by

$$(L_\lambda \vec{u})_\alpha := \partial_j(a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta) = \Delta u_\alpha + \lambda\,\partial_\alpha(\mathrm{div}\,\vec{u}), \qquad 1 \le \alpha \le n. \tag{2.32}$$

Next, assuming that $\lambda \in \mathbb{R}$ and $\vec{u}$, π are sufficiently nice functions in a Lipschitz domain $\Omega \subset \mathbb{R}^n$ with outward unit normal ν, define the conormal derivative

$$\begin{aligned} \partial^\lambda_\nu(\vec{u},\pi) &:= \Big(\nu_j a^{\alpha\beta}_{jk}(\lambda)\partial_k u_\beta - \nu_\alpha \pi\Big)_{1\le\alpha\le n} \\ &= \Big[(\nabla\vec{u})^\top + \lambda(\nabla\vec{u})\Big]\nu - \pi\nu \quad \text{on } \partial\Omega, \end{aligned} \tag{2.33}$$

where $\nabla\vec{u} = (\partial_k u_j)_{1\le j,k\le n}$ denotes the Jacobian matrix of the vector-valued function $\vec{u}$, and $\top$ stands for transposition of matrices. Introducing the bilinear form

$$A_\lambda(\xi,\zeta) := a^{\alpha\beta}_{jk}(\lambda)\xi^\alpha_j\zeta^\beta_k, \qquad \forall\,\xi,\,\zeta\ n\times n \text{ matrices}, \tag{2.34}$$

we then have the following useful integration by parts formula:

$$\int_\Omega \langle L_\lambda\vec{u} - \nabla\pi, \vec{w}\rangle\,dx = \int_{\partial\Omega}\langle\partial^\lambda_\nu(\vec{u},\pi),\vec{w}\rangle\,d\sigma - \int_\Omega\Big\{A_\lambda(\nabla\vec{u},\nabla\vec{w}) - \pi(\mathrm{div}\,\vec{w})\Big\}\,dx. \tag{2.35}$$

In turn, this readily implies that

$$\begin{aligned} \int_\Omega \langle L_\lambda\vec{u} - \nabla\pi,\vec{w}\rangle\,dx - \int_\Omega\langle L_\lambda\vec{w} - \nabla\rho,\vec{u}\rangle\,dx &= \int_{\partial\Omega}\Big\{\langle\partial^\lambda_\nu(\vec{u},\pi),\vec{w}\rangle - \langle\partial^\lambda_\nu(\vec{w},\rho),\vec{u}\rangle\Big\}\,d\sigma \\ &\quad + \int_\Omega\Big\{\pi(\mathrm{div}\,\vec{w}) - \rho(\mathrm{div}\,\vec{u})\Big\}\,dx. \end{aligned} \tag{2.36}$$

Above, it is implicitly assumed that the functions involved are reasonably behaved near the boundary. Such considerations are going to be paid appropriate attention to in each specific application of these integration by parts formulas.

3. Conormal derivative in Besov-Triebel-Lizorkin spaces

For $0 < p, q \leq \infty$ and $s \in \mathbb{R}$, we denote the Besov and Triebel-Lizorkin scales in $\mathbb{R}^n$ by $B^{p,q}_s(\mathbb{R}^n)$ and $F^{p,q}_s(\mathbb{R}^n)$, respectively (cf., e.g., [42]). Next, given $\Omega \subset \mathbb{R}^n$ Lipschitz domain and $0 < p, q \leq \infty$, $\alpha \in \mathbb{R}$, we set

$$\begin{aligned} A^{p,q}_{\alpha}(\Omega) &:= \{u \in \mathscr{D}'(\Omega) : \exists\, v \in A^{p,q}_{\alpha}(\mathbb{R}^n) \text{ with } v|_{\Omega} = u\}, \\ A^{p,q}_{\alpha,0}(\Omega) &:= \{u \in A^{p,q}_{\alpha}(\mathbb{R}^n) \text{ with } \operatorname{supp} u \subseteq \overline{\Omega}\}, \\ A^{p,q}_{\alpha,z}(\Omega) &:= \{u|_{\Omega} : u \in A^{p,q}_{\alpha,0}(\Omega)\}, \end{aligned} \tag{3.1}$$

where $A \in \{B, F\}$. Finally, we let $B^{p,q}_s(\partial\Omega)$ stand for the Besov class on the Lipschitz manifold $\partial\Omega$, obtained by transporting (via a partition of unity and pull-back) the standard scale $B^{p,q}_s(\mathbb{R}^{n-1})$. We shall frequently use the abbreviation

$$L^p_s(\Omega) := F^{p,2}_s(\Omega), \qquad 1 < p < \infty, \;\; s \in \mathbb{R}. \tag{3.2}$$

As is well-known, this is consistent with (2.2) and (2.4).

The existence of a universal linear extension operator, from Lipschitz domains to the entire Euclidean space, which preserves smoothness both on the Besov and the Triebel-Lizorkin scales has been established by V. Rychkov. In [37], he proved the following:

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and denote by $\mathscr{R}_\Omega u := u|_\Omega$ the operator of restriction to Ω. Then there exists a linear, continuous operator E_Ω, mapping distributions in Ω into tempered distributions in $\mathbb{R}^n$, such that whenever $0 < p, q \leq +\infty$, $s \in \mathbb{R}$, then*

$$E_\Omega : A^{p,q}_s(\Omega) \longrightarrow A^{p,q}_s(\mathbb{R}^n) \;\text{ boundedly, satisfying } \mathscr{R}_\Omega \circ E_\Omega f = f, \;\; \forall\, f \in A^{p,q}_s(\Omega), \tag{3.3}$$

for $A = B$ or $A = F$, in the latter case assuming $p < \infty$.

Let us also record here a useful lifting result for fractional order Sobolev spaces on Lipschitz domains, which has been proved in [27].

Proposition 3.2. *Let $1 < p < \infty$ and $s \in \mathbb{R}$. Then for any distribution u in the Lipschitz domain $\Omega \subset \mathbb{R}^n$, the following implication holds:*

$$\nabla u \in L^p_{s-1}(\Omega)^n \implies u \in L^p_s(\Omega). \tag{3.4}$$

The following useful consequence of Proposition 3.2 (cf. [32] for a direct proof) will be used frequently in this paper.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a Lipschitz domain and suppose that $1 < p < \infty$. Then there exists a finite constant $C > 0$ depending only on n, p, and the Lipschitz character of Ω such that every distribution $u \in L^p_{-1}(\Omega)$ with $\nabla u \in L^p_{-1}(\Omega)^n$ has the property that $u \in L^p(\Omega)$ and*

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p_{-1}(\Omega)^n} + C \operatorname{diam}(\Omega)\, \|u\|_{L^p_{-1}(\Omega)} \tag{3.5}$$

holds.

Concerning $\mathscr{R}_\Omega$, the restriction to Ω, let us point out that

$$\mathscr{R}_\Omega : L^p_{s,0}(\Omega) \longrightarrow L^p_{s,z}(\Omega), \quad 1 < p < \infty, \;\; s \in \mathbb{R}, \tag{3.6}$$

is a linear, bounded, onto operator. This permits the factorization

$$L^p_{s,0}(\Omega) \xrightarrow{\ \mathrm{pr}\ } \frac{L^p_{s,0}(\Omega)}{\{u \in L^p_s(\mathbb{R}^n) : \operatorname{supp} u \subseteq \partial\Omega\}} \xrightarrow{\ \mathscr{R}_\Omega\ } L^p_{s,z}(\Omega), \quad 1 < p < \infty, \;\; s \in \mathbb{R}, \tag{3.7}$$

where the first arrow is the canonical projection onto the factor space, and the second arrow is an isomorphism. Moreover, since

$$1 < p < \infty, \ -1 + 1/p < s \Longrightarrow \{u \in L^p_s(\mathbb{R}^n) : \operatorname{supp} u \subseteq \partial\Omega\} = 0 \tag{3.8}$$

then

$$\mathscr{R}_\Omega : L^p_{s,0}(\Omega) \longrightarrow L^p_{s,z}(\Omega) \quad \text{isomorphically if } \ 1 < p < \infty, \ s > -1 + 1/p. \tag{3.9}$$

In this latter case, its inverse is the operator of extension by zero outside Ω, denoted by tilde, i.e.,

$$L^p_{s,z}(\Omega) \ni u \longmapsto \widetilde{u} \in L^p_{s,0}(\Omega), \qquad 1 < p < \infty, \quad -1 + 1/p < s. \tag{3.10}$$

In particular, this allows the identification

$$L^p_{s,0}(\Omega) \equiv L^p_{s,z}(\Omega), \qquad \forall\, p \in (1, \infty), \quad \forall\, s > -1 + 1/p. \tag{3.11}$$

Let us also point out that, if $1 < p < \infty$ and $s \in \mathbb{R}$, we have the continuous embedding

$$L^p_{s,z}(\Omega) \hookrightarrow L^p_s(\Omega) \tag{3.12}$$

and, in fact,

$$L^p_s(\Omega) = L^p_{s,z}(\Omega) \ \text{ if } \ s < \tfrac{1}{p} \ \text{ and } \ \tfrac{1}{p} - s \notin \mathbb{N}. \tag{3.13}$$

Moreover, for every $j = \{1, ..., n\}$

$$\begin{aligned} \partial_j &: L^p_s(\Omega) \longrightarrow L^p_{s-1}(\Omega), \\ \partial_j &: L^p_{s,0}(\Omega) \longrightarrow L^p_{s-1,0}(\Omega), \\ \partial_j &: L^p_{s,z}(\Omega) \longrightarrow L^p_{s-1,z}(\Omega), \end{aligned} \tag{3.14}$$

are well-defined, linear, bounded operators.

Later on, we shall need duality results for the scales introduced at the beginning of this section. Throughout, all duality pairings on Ω are extensions of the natural pairing between test functions and distributions on Ω. As far as the nature of the dual of $L^p_s(\Omega)$ is concerned, when $1 < p, p' < \infty$, $1/p + 1/p' = 1$ and $s \in \mathbb{R}$ we have that

$$\begin{aligned} &\widetilde{\mathscr{C}^\infty_c(\Omega)} \ni \tilde{\varphi} \longmapsto \varphi \in \mathscr{C}^\infty_c(\Omega) \text{ extends to} \\ &\text{an isomorphism } \Psi : L^p_{s,0}(\Omega) \longrightarrow \Big(L^{p'}_{-s}(\Omega)\Big)^*. \end{aligned} \tag{3.15}$$

Above, tilde denotes the extension by zero outside Ω, and the extension in question is achieved via density, as the inclusions

$$\widetilde{\mathscr{C}^\infty_c(\Omega)} \hookrightarrow L^p_{s,0}(\Omega), \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{3.16}$$

$$\mathscr{C}^\infty_c(\Omega) \hookrightarrow \Big(L^p_s(\Omega)\Big)^*, \quad 1 < p < \infty, \ s \in \mathbb{R}, \tag{3.17}$$

have dense ranges. In what follows, we shall frequently identify the spaces $L^p_{s,0}(\Omega)$ and $\Big(L^{p'}_{-s}(\Omega)\Big)^*$ without making any specific reference to the isomorphism Ψ in (3.15). For example, we shall write that

$$\begin{aligned} {}_{L^p_s(\mathbb{R}^n)}\big\langle u, v \big\rangle_{L^{p'}_{-s}(\mathbb{R}^n)} &= {}_{L^p_s(\Omega)}\big\langle \mathscr{R}_\Omega u, v \big\rangle_{L^{p'}_{-s,0}(\Omega)}, \\ \forall\, u \in L^p_s(\mathbb{R}^n), &\qquad \forall\, v \in L^{p'}_{-s,0}(\Omega). \end{aligned} \tag{3.18}$$

Other duality results of interest are

$$\Big(L^p_{s,z}(\Omega)\Big)^* = L^{p'}_{-s}(\Omega) \quad \text{if} \quad 1 < p < \infty \quad \text{and} \quad s > -1 + \tfrac{1}{p}, \tag{3.19}$$

and

$$\Big(L^p_{s}(\Omega)\Big)^* = L^{p'}_{-s,z}(\Omega) \quad \text{if} \quad 1 < p < \infty \quad \text{and} \quad s < \tfrac{1}{p}. \tag{3.20}$$

In particular, if $1 < p < \infty$,

$$\Big(L^p_{s}(\Omega)\Big)^* = L^{p'}_{-s}(\Omega), \qquad \forall\, s \in (-1 + 1/p, 1/p). \tag{3.21}$$

The duality in (3.19) is related to the duality in (3.15) via

$$s > -1 + 1/p \Longrightarrow \begin{cases} {}_{L^p_{s,0}(\Omega)}\Big\langle u, v\Big\rangle_{L^{p'}_{-s}(\mathbb{R}^n)} = {}_{L^p_{s,z}(\Omega)}\Big\langle \mathscr{R}_\Omega u\,,\, \mathscr{R}_\Omega v\Big\rangle_{L^{p'}_{-s}(\Omega)}, \\ \forall\, u \in L^p_{s,0}(\Omega), \qquad \forall\, v \in L^{p'}_{-s}(\mathbb{R}^n). \end{cases} \tag{3.22}$$

As a consequence,

$$s > -1 + 1/p \Longrightarrow \begin{cases} {}_{L^p_{s,0}(\Omega)}\Big\langle \tilde{u}, w\Big\rangle_{L^{p'}_{-s}(\mathbb{R}^n)} = {}_{L^p_{s,z}(\Omega)}\Big\langle u\,,\, \mathscr{R}_\Omega w\Big\rangle_{L^{p'}_{-s}(\Omega)}, \\ \forall\, u \in L^p_{s,z}(\Omega), \qquad \forall\, w \in L^{p'}_{-s}(\mathbb{R}^n), \end{cases} \tag{3.23}$$

and, hence,

$$-1 + 1/p < s < 1/p \Longrightarrow \begin{cases} {}_{L^p_{s,0}(\Omega)}\Big\langle \tilde{u}, w\Big\rangle_{L^{p'}_{-s}(\mathbb{R}^n)} = {}_{L^p_{s}(\Omega)}\Big\langle u, \mathscr{R}_\Omega w\Big\rangle_{L^{p'}_{-s}(\Omega)}, \\ \forall\, u \in L^p_{s}(\Omega), \qquad \forall\, w \in L^{p'}_{-s}(\mathbb{R}^n). \end{cases} \tag{3.24}$$

See the discussion in [29].

Moving on, we shall need a refinement of (2.8) in the context of of Besov and Triebel-Lizorkin spaces (cf. [16], [25]). To state this result, let $(a)_+ := \max\{a, 0\}$.

Proposition 3.4. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and assume that the indices p, s satisfy $\frac{n-1}{n} < p \leq \infty$ and $(n-1)(\frac{1}{p} - 1)_+ < s < 1$. Then the following hold:*

(i) The restriction to the boundary extends to a linear, bounded operator

$$\mathrm{Tr} : B^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,q}_{s}(\partial\Omega) \quad \textit{for} \quad 0 < q \leq \infty. \tag{3.25}$$

For this range of indices, Tr *is onto and has a bounded right inverse*

$$\mathrm{Ex} : B^{p,q}_{s}(\partial\Omega) \longrightarrow B^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{3.26}$$

As far as the null-space of (3.25) *is concerned, if $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ and $0 < q < \infty$, then*

$$B^{p,q}_{s+1/p,z}(\Omega) = \Big\{u \in B^{p,q}_{s+1/p}(\Omega) :\, \mathrm{Tr}\, u = 0\Big\}, \tag{3.27}$$

and

$$\mathscr{C}^\infty_c(\Omega) \hookrightarrow B^{p,q}_{s+1/p,z}(\Omega) \quad \textit{densely}. \tag{3.28}$$

(ii) Similar considerations hold for

$$\mathrm{Tr} : F^{p,q}_{s+\frac{1}{p}}(\Omega) \longrightarrow B^{p,p}_{s}(\partial\Omega) \tag{3.29}$$

with the convention that $q = \infty$ *if* $p = \infty$. *More specifically,* Tr *in* (3.29) *is a linear, bounded, operator which has a linear, bounded right inverse*

$$\mathrm{Ex} : B^{p,p}_{s}(\partial\Omega) \longrightarrow F^{p,q}_{s+\frac{1}{p}}(\Omega). \tag{3.30}$$

Also, if $\frac{n-1}{n} < p < \infty$, $(n-1)(1/p-1)_+ < s < 1$ *and* $\min\{1,p\} \le q < \infty$, *then*

$$F^{p,q}_{s+1/p,z}(\Omega) = \Big\{u \in F^{p,q}_{s+1/p}(\Omega) : \, \mathrm{Tr}\, u = 0\Big\}, \tag{3.31}$$

and

$$\mathscr{C}^\infty_c(\Omega) \hookrightarrow F^{p,q}_{s+1/p,z}(\Omega) \quad \textit{densely}. \tag{3.32}$$

Let X be a Banach space with dual X^*. For every $n \times n$ matrix $F = (F^\alpha_j)_{\alpha,j}$ with entries from X, every $n \times n$ matrix $G = (G^\beta_k)_{\beta,k}$ with entries from X^*, and each $\lambda \in \mathbb{R}$, we set

$$\mathbb{A}^X_\lambda(F,G) := a^{\alpha\beta}_{jk}(\lambda) \; {}_X\langle F^\alpha_j, G^\beta_k\rangle_{X^*}, \tag{3.33}$$

where ${}_X\langle\cdot,\cdot\rangle_{X^*}$ is the duality pairing between X and X^*, and $a^{\alpha\beta}_{jk}(\lambda)$ are as in (2.31). In the sequel, our notation will not emphasize the dependence of $\langle\cdot,\cdot\rangle$ and $\mathbb{A}_\lambda$ on X; however, the particular nature of X should be clear from the context in each case.

Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a bounded Lipschitz domain and assume that $1 < p,q < \infty$, $0 < s < 1$. If $\vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega)^n$, $\pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega)$ and $\vec{f} \in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)^n$ are such that $\Delta\vec{u} - \nabla\pi = \vec{f}|_\Omega$ in Ω, then as suggested by (2.36), it is natural to define

$$\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}} \in B^{p,q}_{s-1}(\partial\Omega)^n = \Big(B^{p',q'}_{1-s}(\partial\Omega)^n\Big)^*,$$
$$1/p + 1/p' = 1, \;\; 1/q + 1/q' = 1, \;\; \lambda \in \mathbb{R}, \tag{3.34}$$

by setting

$$\begin{aligned} \Big\langle \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}}, \vec{\psi}\Big\rangle \;\; &:= \;\; \Big\langle \vec{f}\big|_\Omega, \mathrm{Ex}(\vec{\psi})\Big\rangle + \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla \mathrm{Ex}(\vec{\psi})\Big) \\ &\quad -\Big\langle \pi, \mathrm{div}\, \mathrm{Ex}(\vec{\psi})\Big\rangle, \qquad \forall\, \vec{\psi} \in B^{p',q'}_{1-s}(\partial\Omega)^n, \end{aligned} \tag{3.35}$$

where Ex is the extension operator introduced in Proposition 3.4. The conditions on the indices p,q,s ensure that all duality pairings in the right-hand side of (3.35) are well-defined. Similar considerations apply to the case when $\vec{u},\pi,\vec{f}$ belong to appropriate Triebel-Lizorkin spaces (in which case the conormal $\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}}$ belongs to a suitable diagonal boundary Besov space).

Remark 3.5. Since the conormal $\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}}$ has been defined for a class of (triplets of) functions $\vec{u},\pi,\vec{f}$ for which the expression $\Big[(\nabla\vec{u})^\top + \lambda(\nabla\vec{u})\Big]\nu - \pi\nu$ is, in the standard sense of the trace theory, utterly ill-defined on $\partial\Omega$, it is appropriate to remark that $(\vec{u},\pi,\vec{f}) \mapsto \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}}$ is not an extension of the operation $(\vec{u},\pi) \mapsto \mathrm{Tr}\,\Big[(\nabla\vec{u})^\top + \lambda(\nabla\vec{u})\Big]\nu - \mathrm{Tr}\,\pi\,\nu$ in an ordinary sense. In fact, it is more appropriate to regard the former as a "re-normalization" of the latter trace, in a fashion that depends strongly on the choice of $\vec{f} \in A^{p,q}_{s+1/p-2,0}(\Omega)^n$ as an extension of $\Delta\,\vec{u} - \nabla\pi \in A^{p,q}_{s+1/p-2,z}(\Omega)^n$.

To further shed light on this issue, recall that, for $\vec{u} \in L^2_1(\Omega)^n$, $\pi \in L^2(\Omega)$, $\Delta\vec{u} - \nabla\pi$ is naturally defined as a linear functional in $(L^2_{1,0}(\Omega)^n)^*$. The choice of $\vec{f}$ is the choice of an extension of this linear functional to a functional in $(L^2_1(\Omega)^n)^* = L^2_{-1,0}(\Omega)^n$. As an example, consider $\vec{u} \in L^2_1(\Omega)^n$, $\pi \in L^2(\Omega)$, and suppose that actually $\vec{u} \in L^2_2(\Omega)^n$, $\pi \in L^2_1(\Omega)$ so $\mathrm{Tr}\left[(\nabla\vec{u})^\top + \lambda(\nabla\vec{u})\right]\nu - \mathrm{Tr}\,\pi\,\nu$ is well defined in $L^2(\partial\Omega)^n$. In this case, $\Delta\vec{u} - \nabla\pi \in L^2(\Omega)^n$ has a "natural" extension $\vec{f_0} \in L^2_{-1,0}(\Omega)^n$ (i.e., $\vec{f_0}$ is the extension of $\Delta\vec{u} - \nabla\pi$ to $\mathbb{R}^n$ by setting this equal zero outside Ω). Any other extension $\vec{f_1} \in L^2_{-1,0}(\Omega)^n$ differs from $\vec{f_0}$ by a distribution $\vec{\eta} \in L^2_{-1}(\mathbb{R}^n)^n$ supported on $\partial\Omega$. As is well-known, the space of such distributions is nontrivial. In fact, we have

$$\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f_0}} = \mathrm{Tr}\left[(\nabla\vec{u})^\top + \lambda(\nabla\vec{u})\right]\nu - \mathrm{Tr}\,\pi\,\nu \qquad \text{in } \ L^2(\partial\Omega)^n, \tag{3.36}$$

but if $\vec{\eta} \neq 0$ then $\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f_0}}$ is not equal to $\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f_1}}$. Indeed, by linearity we have that $\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f_1}} = \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f_0}} + \partial^\lambda_\nu(\vec{0},0)_{\vec{\eta}}$ and (3.35) shows that

$$\left\langle \partial^\lambda_\nu(\vec{0},0)_{\vec{\eta}}, \vec{\psi}\right\rangle = \left\langle \vec{\eta}, \mathrm{Ex}(\vec{\psi})\right\rangle \tag{3.37}$$

for every $\vec{\psi} \in L^2_{1/2}(\partial\Omega)^n$. Consequently, $\partial^\lambda_\nu(\vec{0},0)_{\vec{\eta}} \neq 0$ if $\vec{\eta} \neq 0$.

We continue by registering an natural integration by parts formula, which builds on the definition of the "renormalized" conormal (3.35).

Proposition 3.6. *Assume that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain. Fix $s \in (0,1)$, as well as $1 < p, q < \infty$, and denote by p', q' the Hölder conjugate exponents of p and q, respectively.*

Next, suppose that $\vec{w} \in A^{p'q'}_{1-s+1/p'}(\Omega)^n$, $\vec{u} \in A^{p,q}_{s+\frac{1}{p}}(\Omega)^n$, $\pi \in A^{p,q}_{s+\frac{1}{p}-1}(\Omega)$ and $\vec{f} \in A^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)^n$ are such that $\Delta\vec{u} - \nabla\pi = \vec{f}|_\Omega$ in Ω (where, as usual, $A \in \{B,F\}$). Then, for every $\lambda \in \mathbb{R}$, the following integration by parts formula holds:

$$\left\langle \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}}\,,\, \mathrm{Tr}\,\vec{w}\right\rangle = \left\langle \vec{f}\big|_\Omega, \vec{w}\right\rangle + \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\vec{w}\Big) - \left\langle \pi, \mathrm{div}\,\vec{w}\right\rangle. \tag{3.38}$$

Proof. By linearity, it suffices to show that

$$\left\langle \vec{f}\big|_\Omega, \vec{w}\right\rangle + \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\vec{w}\Big) - \left\langle \pi, \mathrm{div}\,\vec{w}\right\rangle = 0 \tag{3.39}$$

if $\vec{w}, \vec{u}, \pi, \vec{f}$ are as in the statement of the proposition and, in addition, $\mathrm{Tr}\,\vec{w} = 0$. Note that the latter condition entails that $\vec{w} \in A^{p',q'}_{1-s,z}(\Omega)^n$ by (3.27), (3.31). Thus, by (3.28), (3.32), $\vec{w}$ can be approximated in $A^{p',q'}_{1-s,z}(\Omega)^n$ by a sequence of vector fields $\vec{w}_j \in \mathscr{C}^\infty_c(\Omega)^n$. Since, thanks to the fact that $\Delta\vec{u} - \nabla\pi = \vec{f}|_\Omega$ as distributions in Ω, we have

$$\left\langle \vec{f}\big|_\Omega, \vec{w}_j\right\rangle + \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\vec{w}_j\Big) - \left\langle \pi, \mathrm{div}\,\vec{w}_j\right\rangle = 0, \qquad j \in \mathbb{N}, \tag{3.40}$$

we can obtain (3.39) by letting $j \to \infty$. □

In order to continue, we introduce the following adaptations of the Besov and Triebel-Lizorkin scales to the Stokes operator

$$\mathcal{B}^{p,q}_s(\Omega) \ := \ \Big\{(\vec{u},\pi,\vec{f}) \in B^{p,q}_{s+\frac{1}{p}}(\Omega)^n \oplus B^{p,q}_{s+\frac{1}{p}-1}(\Omega) \oplus B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)^n : \ \Delta\vec{u} - \nabla\pi = \vec{f}|_\Omega\Big\}, \tag{3.41}$$

and

$$\mathcal{F}^{p,q}_s(\Omega) \ := \ \Big\{(\vec{u},\pi,\vec{f}) \in F^{p,q}_{s+\frac{1}{p}}(\Omega)^n \oplus F^{p,q}_{s+\frac{1}{p}-1}(\Omega) \oplus F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)^n : \ \Delta\vec{u} - \nabla\pi = \vec{f}|_\Omega\Big\}. \tag{3.42}$$

Corollary 3.7. *Suppose that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain, and assume that $s \in (0,1)$, $1 < p, q < \infty$, $1/p + 1/p' = 1/q + 1/q' = 1$. Then*

$$\begin{aligned} \langle \vec{f}\big|_{\Omega}, \vec{w}\rangle - \langle \vec{g}\big|_{\Omega}, \vec{u}\rangle &= \Big\langle \partial^{\lambda}_{\nu}(\vec{u},\pi)_{\vec{f}}\,,\, \mathrm{Tr}\,\vec{w}\Big\rangle - \Big\langle \partial^{\lambda}_{\nu}(\vec{w},\rho)_{\vec{g}}\,,\, \mathrm{Tr}\,\vec{u}\Big\rangle \\ &\quad + \Big\langle \pi, \mathrm{div}\,\vec{w}\Big\rangle - \Big\langle \rho, \mathrm{div}\,\vec{u}\Big\rangle \end{aligned} \tag{3.43}$$

provided either

$$(\vec{u},\pi,\vec{f}) \in \mathcal{B}^{p,q}_{s}(\Omega), \qquad (\vec{w},\rho,\vec{g}) \in \mathcal{B}^{p',q'}_{1-s}(\Omega), \tag{3.44}$$

or

$$(\vec{u},\pi,\vec{f}) \in \mathcal{F}^{p,q}_{s}(\Omega), \qquad (\vec{w},\rho,\vec{g}) \in \mathcal{F}^{p',q'}_{1-s}(\Omega). \tag{3.45}$$

Proof. This follows from (3.38) much as (2.36) follows from (2.35). □

4. Sesquilinear forms and their associated operators

In this section we describe a few basic facts on sesquilinear forms and linear operators associated with them. Throughout, given two Banach spaces $\mathcal{X}$, $\mathcal{Y}$, we denote by $\mathcal{B}(\mathcal{X},\mathcal{Y})$ the space of linear, bounded operators from $\mathcal{X}$ into $\mathcal{Y}$, equipped with the strong operator norm. Also, we let $I_{\mathcal{X}}$ stand for the identity operator on $\mathcal{X}$. Finally, we adopt the convention that if $\mathcal{X}$ is a Banach space then $\mathcal{X}^*$ denotes the *adjoint space* of continuous conjugate linear functionals on $\mathcal{X}$, also known as the *conjugate dual* of $\mathcal{X}$. In this scenario, we let ${}_{\mathcal{X}}\langle\cdot,\cdot\rangle_{\mathcal{X}^*}$ denote the duality pairing between $\mathcal{X}$ and $\mathcal{X}^*$.

Let $\mathcal{H}$ be a complex separable Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ (antilinear in the first and linear in the second argument), $\mathcal{V}$ a reflexive Banach space continuously and densely embedded into $\mathcal{H}$. Then also $\mathcal{H}$ embeds continuously and densely into $\mathcal{V}^*$, i.e.,

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}^* \quad \text{continuously and densely.} \tag{4.1}$$

Here the continuous embedding $\mathcal{H} \hookrightarrow \mathcal{V}^*$ is accomplished via the identification

$$\mathcal{H} \ni u \mapsto (\cdot, u)_{\mathcal{H}} \in \mathcal{V}^*. \tag{4.2}$$

In particular, if the sesquilinear form

$${}_{\mathcal{V}}\langle\cdot,\cdot\rangle_{\mathcal{V}^*} \colon \mathcal{V}\times\mathcal{V}^* \to \mathbb{C} \tag{4.3}$$

denotes the duality pairing between $\mathcal{V}$ and $\mathcal{V}^*$, then

$${}_{\mathcal{V}}\langle u, v\rangle_{\mathcal{V}^*} = (u,v)_{\mathcal{H}}, \quad u\in\mathcal{V}, \quad v \in \mathcal{H}\hookrightarrow\mathcal{V}^*, \tag{4.4}$$

that is, the $\mathcal{V},\mathcal{V}^*$ pairing ${}_{\mathcal{V}}\langle\cdot,\cdot\rangle_{\mathcal{V}^*}$ is compatible with the scalar product $(\cdot,\cdot)_{\mathcal{H}}$ in $\mathcal{H}$.

Let $T \in \mathcal{B}(\mathcal{V},\mathcal{V}^*)$. Since $\mathcal{V}$ is reflexive, i.e. $(\mathcal{V}^*)^* = \mathcal{V}$, one has

$$T\colon \mathcal{V}\to\mathcal{V}^*, \quad T^*\colon \mathcal{V}\to\mathcal{V}^* \tag{4.5}$$

and

$${}_{\mathcal{V}}\langle u, Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle T^*u, v\rangle_{(\mathcal{V}^*)^*} = {}_{\mathcal{V}^*}\langle T^*u, v\rangle_{\mathcal{V}} = \overline{{}_{\mathcal{V}}\langle v, T^*u\rangle_{\mathcal{V}^*}}. \tag{4.6}$$

Self-adjointness of T is then defined as the property that $T = T^*$, that is,

$${}_{\mathcal{V}}\langle u, Tv\rangle_{\mathcal{V}^*} = {}_{\mathcal{V}^*}\langle Tu, v\rangle_{\mathcal{V}} = \overline{{}_{\mathcal{V}}\langle v, Tu\rangle_{\mathcal{V}^*}}, \quad u, v\in\mathcal{V}, \tag{4.7}$$

nonnegativity of T is defined as the demand that

$${}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq 0, \quad u\in\mathcal{V}, \tag{4.8}$$

and *boundedness from below of* T *by* $c \in \mathbb{R}$ is defined as the property that

$$ {}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq c\|u\|^2_{\mathcal{H}}, \qquad \forall\, u \in \mathcal{V}. \tag{4.9} $$

(Note that, by (4.4), this is equivalent to ${}_{\mathcal{V}}\langle u, Tu\rangle_{\mathcal{V}^*} \geq c\,{}_{\mathcal{V}}\langle u, u\rangle_{\mathcal{V}^*}$ for all $u \in \mathcal{V}$.)

Next, let the sesquilinear form $a(\cdot,\cdot)\colon \mathcal{V}\times\mathcal{V} \to \mathbb{C}$ (antilinear in the first and linear in the second argument) be $\mathcal{V}$*-bounded.* That is, there exists a $c_a > 0$ such that

$$ |a(u,v)| \leq c_a\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}, \quad u, v \in \mathcal{V}. \tag{4.10} $$

Then $\widetilde{A}$ defined by

$$ \widetilde{A}\colon \begin{cases} \mathcal{V} \to \mathcal{V}^*, \\ v \mapsto \widetilde{A}v = a(\cdot, v), \end{cases} \tag{4.11} $$

satisfies

$$ \widetilde{A} \in \mathcal{B}(\mathcal{V}, \mathcal{V}^*) \ \text{ and } \ {}_{\mathcal{V}}\big\langle u, \widetilde{A}v\big\rangle_{\mathcal{V}^*} = a(u,v), \quad u, v \in \mathcal{V}. \tag{4.12} $$

In the sequel, we shall refer to $\widetilde{A}$ as *the operator induced by the form* $a(\cdot,\cdot)$.

Assuming further that $a(\cdot,\cdot)$ is *symmetric*, that is,

$$ a(u,v) = \overline{a(v,u)}, \qquad u, v \in \mathcal{V}, \tag{4.13} $$

and that a is $\mathcal{V}$*-coercive*, that is, there exists a constant $C_0 > 0$ such that

$$ a(u,u) \geq C_0\|u\|^2_{\mathcal{V}}, \qquad u \in \mathcal{V}, \tag{4.14} $$

respectively, then,

$$ \widetilde{A}\colon \mathcal{V} \to \mathcal{V}^* \ \text{ is bounded, self-adjoint, and boundedly invertible.} \tag{4.15} $$

Moreover, denoting by A the part of $\widetilde{A}$ in $\mathcal{H}$, defined by

$$ D(A) := \big\{u \in \mathcal{V} : \ \widetilde{A}u \in \mathcal{H}\big\} \subseteq \mathcal{H}, \quad A := \widetilde{A}\big|_{D(A)}\colon D(A) \to \mathcal{H}, \tag{4.16} $$

then A is a (possibly unbounded) self-adjoint operator in $\mathcal{H}$ satisfying

$$ A \geq C_0 I_{\mathcal{H}}, \tag{4.17} $$

$$ D\big(A^{1/2}\big) = \mathcal{V}. \tag{4.18} $$

In particular,

$$ A^{-1} \in \mathcal{B}(\mathcal{H}). \tag{4.19} $$

The facts (4.1)–(4.19) are a consequence of the Lax–Milgram theorem and the second representation theorem for symmetric sesquilinear forms. Details can be found, for instance, in [3, §VI.3, §VII.1], [7, Ch. IV], and [23].

Next, consider a symmetric form $b(\cdot,\cdot)\colon \mathcal{V}\times\mathcal{V} \to \mathbb{C}$ and assume that b is *bounded from below by* $c_b \in \mathbb{R}$, that is,

$$ b(u,u) \geq c_b\|u\|^2_{\mathcal{H}}, \qquad u \in \mathcal{V}. \tag{4.20} $$

Introducing the scalar product $(\cdot,\cdot)_{\mathcal{V}(b)}\colon \mathcal{V}\times\mathcal{V} \to \mathbb{C}$ (with associated norm $\|\cdot\|_{\mathcal{V}(b)}$) by

$$ (u,v)_{\mathcal{V}(b)} := b(u,v) + (1 - c_b)(u,v)_{\mathcal{H}}, \qquad u, v \in \mathcal{V}, \tag{4.21} $$

turns $\mathcal{V}$ into a pre-Hilbert space $(\mathcal{V}; (\cdot,\cdot)_{\mathcal{V}(b)})$, which we denote by $\mathcal{V}(b)$. The form b is called *closed* if $\mathcal{V}(b)$ is actually complete, and hence a Hilbert space. The form b is called *closable* if it has a closed extension. If b is closed, then

$$ |b(u,v) + (1 - c_b)(u,v)_{\mathcal{H}}| \leq \|u\|_{\mathcal{V}(b)}\|v\|_{\mathcal{V}(b)}, \quad u, v \in \mathcal{V}, \tag{4.22} $$

and

$$|b(u,u) + (1-c_b)\|u\|_{\mathcal{H}}^2| = \|u\|_{\mathcal{V}(b)}^2, \qquad u \in \mathcal{V}, \tag{4.23}$$

show that the form $b(\cdot,\cdot) + (1-c_b)(\cdot,\cdot)_{\mathcal{H}}$ is a symmetric, $\mathcal{V}$-bounded, and $\mathcal{V}$-coercive sesquilinear form.

Hence, by (4.11) and (4.12), there exists a linear map

$$\widetilde{B}_{c_b} \colon \begin{cases} \mathcal{V}(b) \to \mathcal{V}(b)^*, \\ \quad v \mapsto \widetilde{B}_{c_b} v := b(\cdot, v) + (1-c_b)(\cdot, v)_{\mathcal{H}}, \end{cases} \tag{4.24}$$

with

$$\widetilde{B}_{c_b} \in \mathcal{B}(\mathcal{V}(b), \mathcal{V}(b)^*) \ \text{ and } \ {}_{\mathcal{V}(b)}\big\langle u, \widetilde{B}_{c_b} v \big\rangle_{\mathcal{V}(b)^*} = b(u,v) + (1-c_b)(u,v)_{\mathcal{H}}, \quad u,v \in \mathcal{V}. \tag{4.25}$$

Introducing the linear map

$$\widetilde{B} := \widetilde{B}_{c_b} + (c_b - 1)\widetilde{I} \colon \mathcal{V}(b) \to \mathcal{V}(b)^*, \tag{4.26}$$

where $\widetilde{I} \colon \mathcal{V}(b) \hookrightarrow \mathcal{V}(b)^*$ denotes the continuous inclusion (embedding) map of $\mathcal{V}(b)$ into $\mathcal{V}(b)^*$, one obtains a self-adjoint operator B in $\mathcal{H}$ by restricting $\widetilde{B}$ to $\mathcal{H}$,

$$D(B) = \big\{u \in \mathcal{V} : \widetilde{B}u \in \mathcal{H}\big\} \subseteq \mathcal{H}, \quad B = \widetilde{B}\big|_{D(B)} \colon D(B) \to \mathcal{H}, \tag{4.27}$$

satisfying the following properties:

$$B \geq c_b I_{\mathcal{H}}, \tag{4.28}$$

$$D\big(|B|^{1/2}\big) = D\big((B - c_b I_{\mathcal{H}})^{1/2}\big) = \mathcal{V}, \tag{4.29}$$

$$b(u,v) = \big(|B|^{1/2}u, U_B |B|^{1/2} v\big)_{\mathcal{H}} \tag{4.30}$$

$$\qquad = \big((B - c_b I_{\mathcal{H}})^{1/2}u, (B - c_b I_{\mathcal{H}})^{1/2} v\big)_{\mathcal{H}} + c_b (u,v)_{\mathcal{H}} \tag{4.31}$$

$$\qquad = {}_{\mathcal{V}(b)}\big\langle u, \widetilde{B}v \big\rangle_{\mathcal{V}(b)^*}, \quad u,v \in \mathcal{V}, \tag{4.32}$$

$$b(u,v) = (u, Bv)_{\mathcal{H}}, \quad u \in \mathcal{V},\ v \in D(B), \tag{4.33}$$

$$\begin{aligned} D(B) = \{v \in \mathcal{V} :\ & \text{there exists } f_v \in \mathcal{H} \text{ such that} \\ & b(w,v) = (w, f_v)_{\mathcal{H}} \text{ for all } w \in \mathcal{V}\}, \\ Bu = f_u, \quad & u \in D(B), \end{aligned} \tag{4.34}$$

$$D(B) \text{ is dense in } \mathcal{H} \text{ and in } \mathcal{V}(b). \tag{4.35}$$

Properties (4.34) and (4.35) uniquely determine B. Here U_B in (4.31) is the partial isometry in the polar decomposition of B, that is,

$$B = U_B |B|, \qquad |B| = (B^* B)^{1/2}. \tag{4.36}$$

Definition 4.1. The operator B is called the operator associated with the form $b(\cdot,\cdot)$.

The norm in the Hilbert space $\mathcal{V}(b)^*$ is given by

$$\|\ell\|_{\mathcal{V}(b)^*} = \sup\{{}_{\mathcal{V}(b)}\langle u, \ell\rangle_{\mathcal{V}(b)^*} : \|u\|_{\mathcal{V}(b)} \leq 1\}, \quad \ell \in \mathcal{V}(b)^*, \tag{4.37}$$

with associated scalar product,

$$(\ell_1, \ell_2)_{\mathcal{V}(b)^*} = {}_{\mathcal{V}(b)}\big\langle \big(\widetilde{B} + (1-c_b)\widetilde{I}\big)^{-1}\ell_1, \ell_2 \big\rangle_{\mathcal{V}(b)^*}, \quad \ell_1, \ell_2 \in \mathcal{V}(b)^*. \tag{4.38}$$

Since

$$\big\|\big(\widetilde{B}+(1-c_b)\widetilde{I}\big)v\big\|_{\mathcal{V}(b)^*}=\|v\|_{\mathcal{V}(b)},\quad v\in\mathcal{V},\tag{4.39}$$

the Riesz representation theorem yields

$$\big(\widetilde{B}+(1-c_b)\widetilde{I}\big)\in\mathcal{B}(\mathcal{V}(b),\mathcal{V}(b)^*)\ \text{ and }\ \big(\widetilde{B}+(1-c_b)\widetilde{I}\big)\colon\mathcal{V}(b)\to\mathcal{V}(b)^*\ \text{ is unitary.}\tag{4.40}$$

In addition,

$$\begin{aligned}{}_{\mathcal{V}(b)}\big\langle u,\big(\widetilde{B}+(1-c_b)\widetilde{I}\big)v\big\rangle_{\mathcal{V}(b)^*}&=\Big(\big(B+(1-c_b)I_{\mathcal{H}}\big)^{1/2}u,\big(B+(1-c_b)I_{\mathcal{H}}\big)^{1/2}v\Big)_{\mathcal{H}}\\&=(u,v)_{\mathcal{V}(b)},\quad u,v\in\mathcal{V}(b).\end{aligned}\tag{4.41}$$

In particular,

$$\big\|(B+(1-c_b)I_{\mathcal{H}})^{1/2}u\big\|_{\mathcal{H}}=\|u\|_{\mathcal{V}(b)},\quad u\in\mathcal{V}(b),\tag{4.42}$$

and hence

$$(B+(1-c_b)I_{\mathcal{H}})^{1/2}\in\mathcal{B}(\mathcal{V}(b),\mathcal{H})\ \text{ and }\ (B+(1-c_b)I_{\mathcal{H}})^{1/2}\colon\mathcal{V}(b)\to\mathcal{H}\ \text{ is unitary.}\tag{4.43}$$

The facts (4.20)–(4.43) comprise the second representation theorem of sesquilinear forms (cf. [7, Sect. IV.2], [10, Sects. 1.2–1.5], [20, Sect. VI.2.6], [34]).

5. Fractional powers and semigroup theory

Assume that $\mathcal{H}$ is a (possibly complex) separable Hilbert space with scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and that $\mathcal{V}$ a reflexive Banach space continuously and densely embedded into $\mathcal{H}$. Also, fix a sesquilinear form $b(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{C}$, which is assumed to be symmetric, nonnegative, bounded, and which satisfies the following coercivity condition: There exist $C_0\in\mathbb{R}$ and $C_1>0$ such that

$$b(u,u)+C_0\|u\|^2_{\mathcal{H}}\geq C_1\|u\|^2_{\mathcal{V}},\qquad u\in\mathcal{V}.\tag{5.1}$$

As a consequence, $\|\cdot\|_{\mathcal{V}(b)}\approx\|\cdot\|_{\mathcal{V}}$. Thus $\mathcal{V}(b)=\mathcal{V}$ and, hence, $b(\cdot,\cdot)$ is also closed.

Let $B:D(B)\subseteq\mathcal{H}\to\mathcal{H}$ be the (possibly unbounded) operator associated with the form $b(\cdot,\cdot)$ as in Definition 4.1. In particular, B is self-adjoint and nonnegative. Also, $tI_{\mathcal{H}}+B$ is invertible on $\mathcal{H}$ for every $t>0$, and $\|t(tI_{\mathcal{H}}+B)^{-1}\|_{\mathcal{B}(\mathcal{H},\mathcal{H})}\leq C$ for $t>0$ (cf., e.g., Proposition 1.22 on p. 13 in [34]). In fact, there exist $\theta\in(0,\pi/2)$ and a finite constant $C>0$ such that $\Sigma_\theta:=\{z\in\mathbb{C}:\ |\arg(z-1)|\leq\theta+\pi/2\}$ is contained in $\mathbb{C}\setminus\operatorname{Spec}(B)$ (where $\operatorname{Spec}(B)$ denotes the spectrum of B as an operator on $\mathcal{H}$) and

$$\|(zI_{\mathcal{H}}+B)^{-1}\|_{\mathcal{B}(\mathcal{H},\mathcal{H})}\leq\frac{C}{1+|z|},\qquad z\in\Sigma_\theta,\tag{5.2}$$

i.e., B is *sectorial.* See, e.g., [4, Theorem 3, p. 374 and Proposition 3, p. 380]. In particular, the operator B generates an analytic semigroup on $\mathcal{H}$ according to the formula

$$e^{zB}u:=\frac{1}{2\pi i}\int_{\Gamma_{\theta'}}e^{-\zeta z}(\zeta I_{\mathcal{H}}+B)^{-1}u\,d\zeta,\qquad|\arg(z)|<\pi/2-\theta',\quad u\in\mathcal{H},\tag{5.3}$$

where $\theta'\in(\theta,\pi/2)$ and $\Gamma_{\theta'}:=\{\pm re^{i\theta'}:\ r>0\}$. Cf. [4] and [35] for a more detailed discussion in this regard.

Moving on, we denote by $\{E_B(\mu)\}_{\mu\in\mathbb{R}}$ the family of spectral projections associated with B, and for each $u\in\mathcal{H}$ introduce the function ρ_u by

$$\rho_u:\mathbb{R}\longrightarrow[0,\infty),\qquad\rho_u(\mu):=(E_B(\mu)u,u)_{\mathcal{H}}=\|E_B(\mu)u\|^2_{\mathcal{H}}.\tag{5.4}$$

Clearly, ρ_u is bounded, non-decreasing, right-continuous, and

$$\lim_{\mu\downarrow-\infty}\rho_u(\mu)=0,\quad \lim_{\mu\uparrow\infty}\rho_u(\mu)=\|u\|_{\mathcal{H}}^2,\qquad \forall\, u\in\mathcal{H}. \tag{5.5}$$

Hence, ρ_u generates a measure, denoted by $d\rho_u$, in a canonical manner. A function $f\colon\mathbb{R}\to\mathbb{C}$ is then called *dE_B-measurable* if it is $d\rho_u$-measurable for each $u\in\mathcal{H}$. As is well-known, all Borel measurable functions are dE_B-measurable functions. For a Borel measurable function $f\colon\mathbb{R}\to\mathbb{C}$ we then define the (possibly) unbounded operator by setting

$$\begin{aligned} &D(f(B)):=\Big\{u\in\mathcal{H}:\ \textstyle\int_{\mathbb{R}}|f|^2\,d\rho_u<+\infty\Big\}\\ &f(B)u:=\textstyle\int_{\mathbb{R}}f(\mu)\,dE_B(\mu)u,\qquad u\in D(f(B)).\end{aligned} \tag{5.6}$$

In particular, for each $\alpha\in[0,1]$, the fractional power B^α of B is a self-adjoint operator

$$B^\alpha : D(B^\alpha)\subset\mathcal{H}\longrightarrow\mathcal{H}. \tag{5.7}$$

Since in our case B is maximally accretive, then so is B^α if $\alpha\in(0,1)$ and for every $u\in D(B)\subset D(B^\alpha)$ we have the representation

$$B^\alpha u=\frac{\sin(\pi\, z)}{\pi}\int_0^\infty t^\alpha B(tI_{\mathcal{H}}+B)^{-1}u\,\frac{dt}{t}. \tag{5.8}$$

See [18], [20]. Other properties are discussed in, e.g., A. Pazy's book [35] and the survey article [1] by W. Arendt, to which we refer the interested reader. Here we only wish to summarize some well-known results of T. Kato and J.-L. Lions (see [19], [23]) which are relevant for our work. Specifically, if B is as above, then

$$D(B^{1/2})=\mathcal{V} \tag{5.9}$$

and, with $[\cdot,\cdot]_\theta$ denoting the complex interpolation bracket,

$$D(B^\theta)=[\mathcal{H},D(B)]_\theta,\quad 0\le\theta\le1. \tag{5.10}$$

Hence, by the reiteration theorem for the complex method, the family

$$\Big\{D(B^{\frac{s}{2}}):\ 0\le s\le2\Big\}\quad\text{is a complex interpolation scale.} \tag{5.11}$$

In particular,

$$D(B^{\theta/2})=[\mathcal{H},\mathcal{V}]_\theta,\quad 0\le\theta\le1. \tag{5.12}$$

We wish to further elaborate on this topic by shedding some light on the nature of $D(B^\alpha)$ when $\alpha\in(1/2,1)$. This requires some preparations. To get started, denote by $\widetilde{B}\in\mathcal{B}(\mathcal{V},\mathcal{V}^*)$ the operator induced by the form $b(\cdot,\cdot)$ (so that B is the part of $\widetilde{B}$ in $\mathcal{H}$), and let $\widetilde{I}$ stand for the inclusion of $\mathcal{V}$ into $\mathcal{V}^*$. It then follows from (5.1) that

$$(\widetilde{I}+\widetilde{B})\in\mathcal{B}(\mathcal{V},\mathcal{V}^*)\quad\text{is an isomorphism.} \tag{5.13}$$

The idea is to find another suitable context in which the operator $\widetilde{I}+\widetilde{B}$ is an isomorphism, and then interpolate between this and (5.13). However, in contrast to what goes on for boundedness, invertibility is not, generally speaking, preserved under interpolation. There are, nonetheless, certain specific settings in which this is true. To discuss such a case recall that, if (X_0,X_1) are a couple of compatible Banach spaces, $X_0\cap X_1$ and X_0+X_1 are equipped, respectively, with the norms

$$\begin{aligned} &\|x\|_{X_0\cap X_1}:=\max\left\{\|x\|_{X_0},\,\|x\|_{X_1}\right\},\qquad\text{and}\\ &\|z\|_{X_0+X_1}=\inf\left\{\|x_0\|_{X_0}+\|x_1\|_{X_1}:\ z=x_0+x_1,\ x_i\in X_i,\ i=0,1\right\}.\end{aligned} \tag{5.14}$$

We have:

Lemma 5.1. *Let (X_0, X_1) and (Y_0, Y_1) be two couples of compatible Banach spaces and assume that $T : X_0 + X_1 \longrightarrow Y_0 + Y_1$ is a linear operator with the property that*

$$T : X_i \longrightarrow Y_i \quad \text{is an isomorphism}, \;\; i = 0, 1. \tag{5.15}$$

In addition, assume that there exist Banach spaces X', Y' such that the inclusions

$$X' \hookrightarrow X_0 \cap X_1, \qquad Y' \hookrightarrow Y_0 \cap Y_1, \tag{5.16}$$

are continuous with dense range, and that

$$T : X' \longrightarrow Y' \quad \text{is an isomorphism.} \tag{5.17}$$

Then the operator

$$T : [X_0, X_1]_\theta \longrightarrow [Y_0, Y_1]_\theta \tag{5.18}$$

is an isomorphism for each $0 \leq \theta \leq 1$.

Proof. Denote by $R_i \in \mathcal{B}(Y_i, X_i)$, $i = 0, 1$, the inverses of T in (5.15). Since the operators R_0 and R_1 coincide as mappings in $\mathcal{B}(Y', X')$, by density they also agree as mappings in $\mathcal{B}(Y_0 \cap Y_1, X_0 \cap X_1)$. It is therefore meaningful to define

$$\begin{aligned} & R : Y_0 + Y_1 \longrightarrow X_0 + X_1, \quad \text{by} \\ & R(y_0 + y_1) := R_0(y_0) + R_1(y_1), \;\; y_i \in Y_i, \;\; i = 0, 1. \end{aligned} \tag{5.19}$$

Then R is a linear operator which belongs to $\mathcal{B}(Y_0, X_0) \cap \mathcal{B}(Y_1, X_1)$. Thus, by the interpolation property, R maps $[Y_0, Y_1]_\theta$ boundedly into $[X_0, X_1]_\theta$ for every $\theta \in [0, 1]$. In this latter context, R provides an inverse for $T : [X_0, X_1]_\theta \longrightarrow [Y_0, Y_1]_\theta$, since $RT = I_{X_0 \cap X_1}$ on $X_0 \cap X_1$, which is a dense subspace of $[X_0, X_1]_\theta$, and $TR = I_{Y_0 \cap Y_1}$ on $Y_0 \cap Y_1$, which is a dense subspace of $[Y_0, Y_1]_\theta$. This proves that the operator in (5.18) is indeed an isomorphism for every $\theta \in [0, 1]$. $\square$

After this preamble, we are ready to present the following.

Proposition 5.2. *With the above assumptions and notation,*

$$D(B^{\frac{1+\theta}{2}}) = (\widetilde{I} + \widetilde{B})^{-1}\Big(D(B^{\frac{1-\theta}{2}})\Big)^* \tag{5.20}$$

for every $0 \leq \theta \leq 1$.

Proof. As already remarked above, the operator $\widetilde{I} + \widetilde{B} : \mathcal{V} \to \mathcal{V}^*$ is boundedly invertible. We claim that

$$\widetilde{I} + \widetilde{B} : D(B) \longrightarrow \mathcal{H} \tag{5.21}$$

is invertible as well, when $D(B)$ is equipped with the graph norm $u \mapsto \|u\|_{\mathcal{H}} + \|Bu\|_{\mathcal{H}}$. Indeed, this operator is clearly well-defined, linear and bounded, since $\widetilde{B}$ coincides with B on $D(B)$. Also, the fact that the operator in (5.13) is one-to-one readily entails that so is (5.21). To see that the operator (5.21) is onto, pick an arbitrary $w \in \mathcal{H} \hookrightarrow \mathcal{V}^*$. It follows from (5.13) that there exists $u \in \mathcal{V} \hookrightarrow \mathcal{H}$ such that $(\widetilde{I} + \widetilde{B})u = w$. In turn, this implies that $\widetilde{B}u = w - u \in \mathcal{H}$ and, hence, $u \in D(B)$. This shows that the operator (5.21) is onto, hence ultimately invertible.

Interpolating between (5.13) and (5.21) then proves (with the help of Lemma 5.1, (5.9)-(5.10), and the duality theorem for the complex method) that the operator

$$\widetilde{I} + \widetilde{B} : D(B^{\frac{1+\theta}{2}}) = [\mathcal{V}, D(B)]_\theta \to [\mathcal{V}^*, \mathcal{H}]_\theta = \Big([\mathcal{H}, \mathcal{V}]_{1-\theta}\Big)^* = \Big(D(B^{\frac{1-\theta}{2}})\Big)^* \tag{5.22}$$

is an isomorphism, for every $0 \leq \theta \leq 1$. From this, (5.20) readily follows. $\square$

6. The definition of the Neumann-Stokes operator

In this section we define the Stokes operator when equipped with Neumann boundary conditions in Lipschitz domains in $\mathbb{R}^n$. Subsequently, in Theorem 6.8, we study the functional analytic properties of this operator. We begin by making the following:

Definition 6.1. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and assume that $\lambda \in \mathbb{R}$ is fixed. Define the Stokes operator with Neumann boundary condition as the unbounded operator

$$B_\lambda : D(B_\lambda) \subset \mathcal{H} \longrightarrow \mathcal{H} \tag{6.1}$$

with domain

$$\begin{aligned} D(B_\lambda) \;\; := \;\; \Big\{ \vec{u} \in \mathcal{V} : \text{ there exists } \pi \in L^2(\Omega) \text{ so that } \vec{f} := -\Delta\vec{u} + \nabla\pi \in \mathcal{H} \\ \text{and such that } \partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}} = 0 \text{ in } L^2_{-1/2}(\partial\Omega)^n \Big\}, \end{aligned} \tag{6.2}$$

and acting according to

$$B_\lambda \vec{u} := -\Delta\vec{u} + \nabla\pi, \qquad \vec{u} \in D(B_\lambda), \tag{6.3}$$

assuming that the pair $(\vec{u},\pi)$ satisfies the requirements in the definition of $D(B_\lambda)$ and where we have identified $\vec{f}$ with its extension by zero outside Ω according to (3.11) (for $p = 2$ and $s = 0$) in the expression $\partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}}$.

As it stands, it is not entirely obvious that the above definition is indeed coherent and our first order of business is to clarify this issue. We do so in a series of lemmas, starting with:

Lemma 6.2. *If the pair $(\vec{u},\pi)$ satisfies the requirements in the definition of $D(B_\lambda)$, then $\Delta\pi = 0$ in Ω.*

Proof. Since the vector fields $\vec{u}$ and $\vec{f} := -\Delta\vec{u} + \nabla\pi$ are both divergence-free, it follows that $\Delta\pi = \operatorname{div}(-\Delta\vec{u} + \nabla\pi) = \operatorname{div}\vec{f} = 0$. ☐

Lemma 6.3. *If $\vec{u} \in D(B_\lambda)$, then there exists a unique scalar function $\pi \in L^2(\Omega)$ such that $\vec{f} := -\Delta\vec{u} + \nabla\pi \in \mathcal{H}$ and $\partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}} = 0$ in $L^2_{-1/2}(\partial\Omega)^n$.*

Proof. Fix a vector field $\vec{u} \in D(B_\lambda)$ and assume that $\pi_j \in L^2(\Omega)$, $j = 1, 2$, are such that

$$\vec{f_j} := -\Delta\vec{u} + \nabla\pi_j \in \mathcal{H} \quad \text{and} \quad \partial_\nu^\lambda(\vec{u},\pi_j)_{\vec{f_j}} = 0 \text{ in } L^2_{-1/2}(\partial\Omega)^n, \text{ for } j = 1, 2. \tag{6.4}$$

Set $\pi := \pi_1 - \pi_2 \in L^2(\Omega)$, and note that

$$\nabla\pi = \vec{f_1} - \vec{f_2} \in \mathcal{H} \hookrightarrow L^2_1(\Omega)^n. \tag{6.5}$$

As a consequence,

$$\pi \in L^2_1(\Omega). \tag{6.6}$$

Next, we employ (3.35) and (6.4) in order to write

$$\begin{aligned} 0 \;\; &= \;\; \Big\langle \partial_\nu^\lambda(\vec{u},\pi_1)_{\vec{f_1}} - \partial_\nu^\lambda(\vec{u},\pi_2)_{\vec{f_2}}, \vec{\psi} \Big\rangle \\ &= \;\; \Big\langle \vec{f_1}, \mathrm{Ex}(\vec{\psi}) \Big\rangle + \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\mathrm{Ex}(\vec{\psi})\Big) - \Big\langle \pi_1, \operatorname{div}\mathrm{Ex}(\vec{\psi}) \Big\rangle \\ &\quad - \Big\langle \vec{f_2}, \mathrm{Ex}(\vec{\psi}) \Big\rangle - \mathbb{A}_\lambda\Big(\nabla\vec{u}, \nabla\mathrm{Ex}(\vec{\psi})\Big) + \Big\langle \pi_2, \operatorname{div}\mathrm{Ex}(\vec{\psi}) \Big\rangle \\ &= \;\; \Big\langle \vec{f_1} - \vec{f_2}, \mathrm{Ex}(\vec{\psi}) \Big\rangle - \Big\langle \pi, \operatorname{div}\mathrm{Ex}(\vec{\psi}) \Big\rangle, \end{aligned} \tag{6.7}$$

for every $\vec{\psi} \in L^2_{1/2}(\partial\Omega)^n$. At this stage, we recall (6.5)-(6.6) in order to transform the last expression in (6.7) into

$$\Big\langle \nabla\pi, \mathrm{Ex}(\vec{\psi})\Big\rangle - \Big\langle \pi, \mathrm{div}\,\mathrm{Ex}(\vec{\psi})\Big\rangle = \Big\langle \mathrm{Tr}\,\pi\,,\,\nu\cdot\vec{\psi}\Big\rangle. \tag{6.8}$$

In concert with (6.7), this shows that

$$\Big\langle (\mathrm{Tr}\,\pi)\,\nu\,,\,\vec{\psi}\Big\rangle = 0 \quad \text{for every} \quad \vec{\psi} \in L^2_{1/2}(\partial\Omega)^n, \tag{6.9}$$

from which we may conclude that

$$\mathrm{Tr}\,\pi = 0 \;\text{ in }\; L^2_{1/2}(\partial\Omega)^n. \tag{6.10}$$

This, (6.6) and Lemma 6.2 amount to saying that $\pi \in L^2_1(\Omega)$ is harmonic and satisfies $\mathrm{Tr}\,\pi = 0$. Thus, $\pi = 0$ in Ω, by the uniqueness for the Dirichlet problem. Hence, $\pi_1 = \pi_2$ in Ω, as desired. □

Remark 6.4. In particular, Lemma 6.3 implies that there is no ambiguity in defining $B_\lambda\vec{u}$ as in (6.3).

Recall now the bilinear form (2.34), and consider

$$b_\lambda(\cdot,\cdot) : \mathcal{V}\times\mathcal{V} \longrightarrow \mathbb{R}, \qquad b_\lambda(\vec{u},\vec{v}) := \int_\Omega A_\lambda(\overline{\nabla\vec{u}}, \nabla\vec{v})\,dx. \tag{6.11}$$

Our goal is to study this sesquilinear form. This requires some prerequisites which we now dispense with. First, the following Korn type estimate has been proved in [32].

Proposition 6.5. *Let Ω be a Lipschitz domain and assume that $1 < p < \infty$. Then there exists a finite constant $C > 0$ which depends only on p and the Lipschitz character of Ω such that*

$$\|\vec{u}\|_{L^p_1(\Omega)^n} \leq C\Big\{\|\nabla\vec{u} + \nabla\vec{u}^\top\|_{L^p(\Omega)^{n^2}} + C\,\mathrm{diam}\,(\Omega)^{-1}\|\vec{u}\|_{L^p(\Omega)^n}\Big\}, \tag{6.12}$$

uniformly for $\vec{u} \in L^p_1(\Omega)^n$.

We shall also need the the following algebraic result from [32] regarding the bilinear form $A_\lambda(\cdot,\cdot)$ from (2.34).

Proposition 6.6. *For every $\lambda \in (-1,1]$ there exists $\kappa_\lambda > 0$ such that for every $n\times n$-matrix ξ*

$$A_\lambda(\xi,\xi) \geq \kappa_\lambda\,|\xi|^2 \;\; \textit{for} \;\; |\lambda| < 1 \quad \textit{and} \quad A_1(\xi,\xi) \geq \kappa_1\,|\xi + \xi^\top|^2. \tag{6.13}$$

The following well-known result (cf. [5]) is also going to be useful shortly.

Lemma 6.7. *Let Ω be an open subset of $\mathbb{R}^n$, and assume that $\vec{v} \in [\mathscr{D}(\Omega)']^n$ is a vector-valued distribution which annihilates $\{\vec{w} \in \mathscr{C}^\infty_c(\Omega)^n : \mathrm{div}\,\vec{w} = 0 \text{ in } \Omega\}$. Then there exists a scalar distribution $q \in \mathscr{D}(\Omega)'$ with the property that $\vec{v} = \nabla q$ in Ω.*

We are now ready to state and prove the main result of this section. Recall the spaces $\mathcal{V}$, $\mathcal{H}$ from (2.10), (2.9), along with the form $b_\lambda(\cdot,\cdot)$ from (6.11).

Theorem 6.8. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and assume that $\lambda \in (-1,1]$ is fixed. Then the sesquilinear form $b_\lambda(\cdot,\cdot)$ introduced in (6.11) is symmetric, bounded, non-negative, and closed.*

Furthermore, the Neumann-Stokes operator B_λ, originally introduced in (6.1)-(6.3)*, is (in the terminology of* § *4) the operator associated with $b_\lambda(\cdot,\cdot)$. As a consequence,*

$$B_\lambda \ \text{ is self-adjoint and nonnegative on } \ \mathcal{H}, \tag{6.14}$$

$$-B_\lambda \ \text{ generates an analytic semigroup on } \ \mathcal{H}, \tag{6.15}$$

$$D(|B_\lambda|^{1/2}) = D(B_\lambda^{1/2}) = \mathcal{V}, \tag{6.16}$$

$$D(B_\lambda) \ \text{ is dense both in } \mathcal{V} \text{ and in } \mathcal{H}. \tag{6.17}$$

Finally, $\mathrm{Spec}(B_\lambda)$*, the spectrum of the operator* (6.1)-(6.3) *is a discreet subset of* $[0,\infty)$*.*

Proof. Lemma 2.1 ensures that (4.1) holds, hence the formalism from § 4 applies. That the form $b_\lambda(\cdot,\cdot)$ in (6.11) is nonnegative, symmetric, sesquilinear and continuous is clear from its definition. In addition, this form is coercive, hence closed. Indeed, when $|\lambda| < 1$ this follows directly from Proposition 6.6, whereas when $\lambda = 1$ this is a consequence of the second inequality in (6.13) and Proposition 6.5.

We next wish to show the coincidence between the domain $D(B_\lambda)$ of the Neumann-Stokes operator in (6.2) and the space

$$\big\{\vec{u} \in \mathcal{V} : \text{ there exists } \vec{f} \in \mathcal{H} \text{ such that } b_\lambda(\vec{w},\vec{u}) = (\vec{w},\vec{f})_{\mathcal{H}} \text{ for all } \vec{w} \in \mathcal{V}\big\}. \tag{6.18}$$

In one direction, fix $\vec{u} \in \mathcal{V}$ such that there exists $\vec{f} \in \mathcal{H}$ for which

$$\int_\Omega A_\lambda(\overline{\nabla\vec{w}}, \nabla\vec{u})\,dx = \int_\Omega \langle\overline{\vec{w}}, \vec{f}\rangle\,dx \quad \text{for every} \quad \vec{w} \in \mathcal{V}. \tag{6.19}$$

Specializing (6.19) to the case when $\vec{w} \in \mathscr{C}_c^\infty(\Omega)^n$ is divergence-free yields, e.g., on account of (2.35) used with $\pi = 0$, that

$$\text{the distribution } \vec{f} + \Delta\vec{u} \text{ vanishes on } \ \Big\{\vec{w} \in \mathscr{C}_c^\infty(\Omega)^n : \operatorname{div}\vec{w} = 0 \text{ in } \Omega\Big\}. \tag{6.20}$$

Then, by virtue of Lemma 6.7, there exists a scalar distribution $\widetilde{\pi}$ in Ω such that

$$\nabla\widetilde{\pi} = \vec{f} + \Delta\vec{u} \in L^2_{-1}(\Omega)^n. \tag{6.21}$$

Going further, (6.21) and Corollary 3.3 imply that, in fact,

$$\widetilde{\pi} \in L^2(\Omega) \quad \text{and} \quad \vec{f} = -\Delta\vec{u} + \nabla\widetilde{\pi} \ \text{ in } \ \Omega. \tag{6.22}$$

At this point we make the claim that there exists a constant $c \in \mathbb{R}$ with the property that

$$\pi := \widetilde{\pi} - c \implies \partial_\nu^\lambda(\vec{u},\pi)_{\vec{f}} = 0 \ \text{ in } \ L^2_{-1/2}(\partial\Omega)^n. \tag{6.23}$$

To justify this, we first note that (3.38) (used with $-\vec{f}$ in place of $\vec{f}$) and (6.19) force

$$\Big\langle \partial_\nu^\lambda(\vec{u},\widetilde{\pi})_{\vec{f}}\,,\, \mathrm{Tr}\,\vec{w}\Big\rangle = 0 \quad \text{for every } \vec{w} \in \mathcal{V}, \tag{6.24}$$

hence, further,

$$\Big\langle \partial_\nu^\lambda(\vec{u},\widetilde{\pi})_{\vec{f}}\,,\, \vec{\varphi}\Big\rangle = 0 \quad \text{for every } \vec{\varphi} \in L^2_{1/2,\nu}(\partial\Omega), \tag{6.25}$$

by Lemma 2.3. To continue, fix some vector field $\vec{\varphi}_o \in L^2_{1/2}(\partial\Omega)^n$ with the property that $\int_{\partial\Omega} \nu\cdot\vec{\varphi}_o\,d\sigma = 1$, and define

$$c := \Big\langle \partial_\nu^\lambda(\vec{u},\widetilde{\pi})_{\vec{f}}\,,\, \vec{\varphi}_o\Big\rangle. \tag{6.26}$$

Now, given an arbitrary $\vec{\varphi} \in L^2_{1/2}(\partial\Omega)^n$, set $\eta := \fint_{\partial\Omega} \nu \cdot \vec{\varphi}\, d\sigma$ and compute

$$\begin{aligned} \Big\langle \partial^\lambda_\nu(\vec{u},\widetilde{\pi})_{\vec{f}}\,,\,\vec{\varphi}\Big\rangle &= \Big\langle \partial^\lambda_\nu(\vec{u},\widetilde{\pi})_{\vec{f}}\,,\,\vec{\varphi}-\eta\vec{\varphi}_o\Big\rangle + \eta\Big\langle \partial^\lambda_\nu(\vec{u},\widetilde{\pi})_{\vec{f}}\,,\,\vec{\varphi}_o\Big\rangle \\ &= 0 + \langle c\,\nu\,,\,\vec{\varphi}\rangle, \end{aligned} \tag{6.27}$$

by (6.25), (6.26) and the definition of λ. Since $\vec{\varphi} \in L^2_{1/2}(\partial\Omega)$ is arbitrary, this proves that

$$\partial^\lambda_\nu(\vec{u},\widetilde{\pi})_{\vec{f}} = c\,\nu \quad \text{in} \quad L^2_{-1/2}(\partial\Omega)^n. \tag{6.28}$$

Thus,

$$\partial^\lambda_\nu(\vec{u},\widetilde{\pi}-c)_{\vec{f}} = \partial^\lambda_\nu(\vec{u},\widetilde{\pi})_{\vec{f}} - \partial^\lambda_\nu(\vec{0},c)_{\vec{0}} = c\,\nu - c\,\nu = 0 \ \text{ in } \ L^2_{-1/2}(\partial\Omega)^n, \tag{6.29}$$

hence (6.23) holds. Note that (6.22) also ensures that $\pi \in L^2(\Omega)$ and $\vec{f} = -\Delta\vec{u} + \nabla\pi$ in Ω. Together, these conditions prove that the space in (6.18) is contained in $D(B_\lambda)$ (defined in (6.2)).

Conversely, the inclusion of $D(B_\lambda)$ into the space in (6.18) is a direct consequence of the definition of the domain of the Neumann-Stokes operator (in (6.2)) and the integration by parts formula (3.38).

Once $D(B_\lambda)$ has been identified with the space in (6.18), the fact that the Neumann-Stokes operator B_λ, in (6.1)-(6.3) is, in the terminology of § 4, the operator associated with the form $b_\lambda(\cdot,\cdot)$ follows from (4.34). Finally, the claim made about $\mathrm{Spec}\,(B_\lambda)$ is a consequence of the fact that B_λ is nonnegative and has a compact resolvent. □

7. The Stokes scale adapted to Neumann boundary conditions

Given a Lipschitz domain $\Omega \subset \mathbb{R}^n$ and $1 < p < \infty$, $s \in \mathbb{R}$, we set

$$V^{s,p}(\Omega) := \Big\{\vec{u} \in L^p_s(\Omega)^n : \operatorname{div}\vec{u} = 0 \text{ in } \Omega\Big\}. \tag{7.1}$$

The first main result of this section is to show that the above scale is stable under complex interpolation.

Theorem 7.1. *For each Lipschitz domain $\Omega \subset \mathbb{R}^n$, the family*

$$\Big\{V^{s,p}(\Omega) : 1 < p < \infty, \ \ s \in \mathbb{R}\Big\} \tag{7.2}$$

is a complex interpolation scale. In other words, if $[\cdot,\cdot]_\theta$ stands for the usual complex interpolation bracket, then

$$\Big[V^{s_0,p_0}(\Omega)\,,\,V^{s_1,p_1}(\Omega)\Big]_\theta = V^{s,p}(\Omega) \tag{7.3}$$

whenever $1 < p_i < \infty$, $s_i \in \mathbb{R}$, $i = 0,1$, $\theta \in [0,1]$, $\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $s := (1-\theta)s_0 + \theta s_1$.

Before turning to the proof of Theorem 7.1, we recall a version of an abstract interpolation result from [24].

Lemma 7.2. *Let X_i, Y_i, $i = 0,1$, be two pairs of Banach spaces such that $X_0 \cap X_1$ is dense in both X_0 and X_1, and similarly for Y_0, Y_1. Let D be a linear operator such that $D : X_i \to Y_i$ boundedly for $i = 0,1$, and consider the following closed subspaces of X_i, $i = 0,1$:*

$$\operatorname{Ker}(D;X_i) := \{u \in X_i : Du = 0\}, \quad i = 0,1. \tag{7.4}$$

Finally, suppose that there exists a continuous linear mapping $G : Y_i \to X_i$ with the property $D \circ G = I$, the identity on Y_i for $i = 0,1$. Then, for each $0 < \theta < 1$,

$$[\operatorname{Ker}(D;X_0), \operatorname{Ker}(D;X_1)]_\theta = \{u \in [X_0,X_1]_\theta : Du = 0\}, \qquad \theta \in (0,1). \tag{7.5}$$

Proof of Theorem 7.1. Denote by Π the harmonic Newtonian potential, i.e., the operator of convolution with the standard fundamental solution for the Laplacian in $\mathbb{R}^n$. Recall the universal extension operator E_Ω from Theorem 3.1. Without loss of generality, we may assume that $E_\Omega u$ is supported in a fixed compact neighborhood of Ω for every distribution u in Ω. Assuming that this is the case, we set

$$\Pi_\Omega := \mathscr{R}_\Omega \circ \Pi \circ E_\Omega, \tag{7.6}$$

where, as before, $\mathscr{R}_\Omega u := u|_\Omega$ is the operator of restriction to Ω. Given that Π is smoothing of order two, it follows that

$$\Pi_\Omega : L^p_s(\Omega) \longrightarrow L^p_{s+2}(\Omega), \qquad 1 < p < \infty, \;\; s \in \mathbb{R}, \tag{7.7}$$

is a well-defined, linear and bounded operator.

Next, fix $p_0, p_1, p, s_0, s_1, s, \theta$ as in the statement of the theorem. We shall implement Lemma 7.2 in which we take

$$X_i := L^{p_i}_{s_i}(\Omega)^n \quad \text{and} \quad Y_i := L^{p_i}_{s_i-1}(\Omega), \qquad i = 0, 1, \tag{7.8}$$

as well as

$$D := \mathrm{div} \qquad \text{and} \qquad G := \nabla \Pi_\Omega. \tag{7.9}$$

Then since

$$D : X_i \longrightarrow Y_i, \qquad G : Y_i \longrightarrow X_i, \qquad i = 0, 1, \tag{7.10}$$

are well-defined, linear and bounded, and since $D \circ G = I$, the identity, the conclusion in Theorem 7.1 follows from Lemma 7.2. $\square$

Our next goal is to identify the duals of the spaces in the Stokes scale introduced in (7.1). As a preamble, we prove the following.

Proposition 7.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$ with outward unit normal ν and assume that $1 < p < \infty$, $-1 + \frac{1}{p} < s < \frac{1}{p}$. Define the mapping*

$$\nu \cdot \; : V^{s,p}(\Omega) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \tag{7.11}$$

by setting

$$\langle \nu \cdot \vec{u}, \phi \rangle := \langle \vec{u}, \nabla \Phi \rangle \tag{7.12}$$

for each $\phi \in \Big(B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)\Big)^* = B^{p',p'}_{-s+\frac{1}{p}}(\partial\Omega)$, where $\Phi \in L^{p'}_{1-s}(\Omega)$ is such that $\mathrm{Tr}\,\Phi = \phi$. Then the above definition is meaningful and the operator (7.11) *is bounded in the sense that*

$$\|\nu \cdot \vec{u}\|_{B^{p,p}_{s-\frac{1}{p}}(\partial\Omega)} \leq C \|\vec{u}\|_{L^p_s(\Omega)^n}, \tag{7.13}$$

for some finite $C = C(\Omega, s, p) > 0$. Finally, the range of the operator (7.11)-(7.12) *is*

$$\Big\{ f \in B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) : \langle f, 1 \rangle = 0 \Big\}. \tag{7.14}$$

Proof. This follows from Proposition 2.7 in [29] and Proposition 2.1 in [28]. $\square$

Theorem 7.4. *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and fix $1 < p < \infty$. Next, for each $-1 + 1/p < s < 1/p$, let*

$$J_{s,p} : V^{s,p}(\Omega) \hookrightarrow L^p_s(\Omega)^n \tag{7.15}$$

be the canonical inclusion, and consider its adjoint

$$J^*_{s,p} : L^{p'}_{-s}(\Omega)^n \longrightarrow \Big(V^{s,p}(\Omega)\Big)^*, \tag{7.16}$$

where $1/p + 1/p' = 1$. *Then the mapping* (7.16) *is onto and its kernel is precisely* $\nabla[L^{p'}_{1-s,z}(\Omega)]$. *In particular,*

$$J^*_{s,p} : \frac{L^{p'}_{-s}(\Omega)^n}{\nabla\Big[L^{p'}_{1-s,z}(\Omega)\Big]} \longrightarrow \Big(V^{s,p}(\Omega)\Big)^* \tag{7.17}$$

is an isomorphism.

Proof. Since $V^{s,p}(\Omega)$ is a closed subspace of $L^p_{s,z}(\Omega)$, Hahn-Banach's theorem immediately gives that the mapping (7.16) is onto. That (7.17) is an isomorphism will then follow as soon as we show that $\operatorname{Ker} J^*_{s,p}$, the null-space of the application (7.16), coincides with $\nabla[L^{p'}_{1-s,z}(\Omega)]$. In one direction, if $\vec{u} \in L^{p'}_{-s}(\Omega)^n = \Big(L^p_s(\Omega)^n\Big)^*$ is such that $J^*_{s,p}(\vec{u}) = 0$, then $\langle \vec{u}, \vec{v}\rangle = 0$ for each $\vec{v} \in V^{s,p}(\Omega)$. Choosing $\vec{v} \in \mathscr{C}^\infty_c(\Omega)^n$ such that $\operatorname{div}\vec{v} = 0$ in Ω shows, on account of Lemma 6.7, that there exists a distribution w in Ω such that $\nabla w = \vec{u}$. Proposition 3.2 then ensures that $w \in L^{p'}_{1-s}(\Omega)$, so that $\vec{u} = \nabla w \in \nabla\Big[L^{p'}_{1-s}(\Omega)\Big]$. There remains to show that, after subtracting a suitable constant from w, this function can be made to have trace zero and, hence, belong to $L^{p'}_{1-s,z}(\Omega)$. To this end, note that for each $\vec{v} \in V^{s,p}(\Omega)$ we have

$$0 = \langle \vec{u}, \vec{v}\rangle = \langle \nabla w, \vec{v}\rangle = \langle \operatorname{Tr} w, \nu \cdot \vec{v}\rangle. \tag{7.18}$$

Then the last claim in Proposition 7.3 shows that $\operatorname{Tr} w$ is a constant, as wanted.

Conversely, if $\vec{u} = \nabla\Phi \in L^{p'}_{-s}(\Omega)^n$ for some $\Phi \in L^{p'}_{1-s,z}(\Omega)$ then Proposition 7.3 allows us to write

$$\langle J^*_{s,p}(\vec{u}), \vec{v}\rangle = \langle \nabla\Phi, \vec{v}\rangle = \langle \operatorname{Tr}\Phi, \nu \cdot \vec{v}\rangle = 0, \tag{7.19}$$

for every $\vec{v} \in V^{s,p}(\Omega)$. Thus, $J^*_{s,p}(\vec{u}) = 0$, finishing the proof of the theorem. □

Theorem 7.5. *For each Lipschitz domain* $\Omega \subset \mathbb{R}^n$ *there exists* $\varepsilon = \varepsilon(\Omega) \in (0, 1]$ *with the following significance. Assume that* $1 < p < \infty$, $-1 + 1/p < s < 1/p$ *and that the pair* $(s, 1/p)$ *satisfies either of the following three conditions:*

$$\begin{aligned} &(I): && 0 < \tfrac{1}{p} < \tfrac{1-\varepsilon}{2} \quad \textit{and} \quad -1 + \tfrac{1}{p} < s < \tfrac{3}{p} - 1 + \varepsilon;\\ &(II): && \tfrac{1-\varepsilon}{2} \leq \tfrac{1}{p} \leq \tfrac{1+\varepsilon}{2} \quad \textit{and} \quad -1 + \tfrac{1}{p} < s < \tfrac{1}{p};\\ &(III): && \tfrac{1+\varepsilon}{2} < \tfrac{1}{p} < 1 \quad \textit{and} \quad -2 + \tfrac{3}{p} - \varepsilon < s < \tfrac{1}{p}. \end{aligned} \tag{7.20}$$

Then

$$L^p_s(\Omega)^n = V^{s,p}(\Omega) \oplus \nabla\Big[L^p_{s+1,z}(\Omega)\Big], \tag{7.21}$$

where the direct sum is topological (in fact, orthogonal when $s = 0$ *and* $p = 2$*). Furthermore, if*

$$\mathbb{P} : L^p_s(\Omega)^n \longrightarrow V^{s,p}(\Omega) \tag{7.22}$$

denotes the projection onto the first summand in the decomposition (7.21)*, then its kernel is* $\nabla\Big[L^p_{s+1,z}(\Omega)\Big]$*. In particular,*

$$\mathbb{P}:\frac{L^p_s(\Omega)^n}{\nabla\Big[L^p_{s+1,z}(\Omega)\Big]}\longrightarrow V^{s,p}(\Omega) \tag{7.23}$$

is an isomorphism. Also, the adjoint of the operator

$$\mathbb{P}_{p,s}:L^p_s(\Omega)^n\xrightarrow{\ \mathbb{P}\ }V^{s,p}(\Omega)\xrightarrow{J_{s,p}}L^p_s(\Omega)^n \tag{7.24}$$

is the operator $\mathbb{P}_{p',-s}$*, where* $1/p+1/p'=1$*, and*

$$\Big(V^{s,p}(\Omega)\Big)^*=V^{-s,p'}(\Omega). \tag{7.25}$$

Proof. The decomposition (7.21) corresponding to the case when $s=0$ has been established in [9] via an approach which reduces matters to the well-posedness of the inhomogeneous Dirichlet problem for the Laplacian in the Lipschitz domain Ω. The more general case considered here can be proved in an analogous fashion. With (7.21) in hand, the claims about the projection (7.22) are straightforward.

Consider next the identification in (7.25). If $\vec{u}\in V^{-s,p'}(\Omega)$ define $\Lambda_{\vec{u}}\in\big(V^{s,p}(\Omega)\big)^*$ by setting

$$\Lambda_{\vec{u}}(\vec{v}):={}_{L^p_s(\Omega)^n}\Big\langle J_{s,p}\vec{v}\,,\,J_{-s,p'}\vec{u}\Big\rangle_{L^{p'}_{-s}(\Omega)^n},\qquad\forall\,\vec{v}\in V^{s,p}(\Omega). \tag{7.26}$$

Note that since $-1+1/p<s<1/p$, the dual of $L^p_s(\Omega)$ is $L^{p'}_{-s}(\Omega)$, hence the duality bracket makes sense. Then the mapping

$$\Phi:V^{-s,p'}(\Omega)\longrightarrow\Big(V^{s,p}(\Omega)\Big)^*,\qquad\Phi(\vec{u}):=\Lambda_{\vec{u}}, \tag{7.27}$$

is well-defined, linear and bounded. Our goal is to show that this is an isomorphism. To prove that Φ is onto, fix $\Lambda\in\big(V^{s,p}(\Omega)\big)^*$. Recall the operator $\mathbb{P}$ from (7.22) and note that $\Lambda\circ\mathbb{P}\in\big(L^p_s(\Omega)\big)^*=L^{p'}_{-s}(\Omega)$. That is, there exists $\vec{w}\in L^{p'}_{-s}(\Omega)$ such that

$$(\Lambda\circ\mathbb{P})\vec{u}={}_{L^p_s(\Omega)^n}\Big\langle\vec{w}\,,\,J_{s,p}\vec{u}\Big\rangle_{L^{p'}_{-s}(\Omega)^n},\qquad\forall\,\vec{u}\in V^{s,p}(\Omega). \tag{7.28}$$

Then $\Lambda_{\mathbb{P}\vec{w}}=\Phi(\mathbb{P}\vec{w})$ satisfies

$$\begin{aligned}\Lambda_{\mathbb{P}\vec{w}}(\vec{v}) &= {}_{L^p_s(\Omega)^n}\Big\langle J_{s,p}\vec{v}\,,\,J_{-s,p'}\mathbb{P}\vec{w}\Big\rangle_{L^{p'}_{-s}(\Omega)^n}={}_{L^p_s(\Omega)^n}\Big\langle J_{s,p}\vec{v}\,,\,\mathbb{P}_{p',-s}\vec{w}\Big\rangle_{L^{p'}_{-s}(\Omega)^n}\\ &= {}_{L^p_s(\Omega)^n}\Big\langle \mathbb{P}_{p,s}J_{s,p}\vec{v}\,,\,\vec{w}\Big\rangle_{L^{p'}_{-s}(\Omega)^n}={}_{L^p_s(\Omega)^n}\Big\langle J_{s,p}\vec{v}\,,\,\vec{w}\Big\rangle_{L^{p'}_{-s}(\Omega)^n}\\ &= (\Lambda\circ\mathbb{P})\vec{v}=\Lambda(\vec{v}),\qquad\forall\,\vec{v}\in V^{s,p}(\Omega).\end{aligned} \tag{7.29}$$

Hence $\Lambda = \Lambda_{\mathbb{P}\vec{w}}$, proving that Φ is onto. To see that Φ is also one-to-one, we note that if $\vec{u} \in V^{-s,p'}(\Omega)$ is such that $\Lambda_{\vec{u}} = 0$, then

$$\begin{aligned}
& {}_{L^p_s(\Omega)^n}\Big\langle J_{s,p}\vec{v}\,,\, J_{-s,p'}\vec{u}\Big\rangle_{L^{p'}_{-s}(\Omega)^n} = 0 \quad \forall\,\vec{v} \in V^{s,p}(\Omega) \\
& \Longrightarrow {}_{L^p_s(\Omega)^n}\Big\langle J_{s,p}\mathbb{P}\vec{w}\,,\, J_{-s,p'}\vec{u}\Big\rangle_{L^{p'}_{-s}(\Omega)^n} = 0 \quad \forall\,\vec{w} \in L^p_s(\Omega) \\
& \Longrightarrow {}_{L^p_s(\Omega)^n}\Big\langle \mathbb{P}_{p,s}\vec{w}\,,\, J_{-s,p'}\vec{u}\Big\rangle_{L^{p'}_{-s}(\Omega)^n} = 0 \quad \forall\,\vec{w} \in L^p_s(\Omega) \\
& \Longrightarrow {}_{L^p_s(\Omega)^n}\Big\langle \vec{w}\,,\, \mathbb{P}_{p',-s}J_{-s,p'}\vec{u}\Big\rangle_{L^{p'}_{-s}(\Omega)^n} = 0 \quad \forall\,\vec{w} \in L^p_s(\Omega) \\
& \Longrightarrow {}_{L^p_s(\Omega)^n}\Big\langle \vec{w}\,,\, J_{-s,p'}\vec{u}\Big\rangle_{L^{p'}_{-s}(\Omega)^n} = 0 \quad \forall\,\vec{w} \in L^p_s(\Omega) \\
& \Longrightarrow J_{-s,p'}\vec{u} = 0 \ \text{ in } \ L^{p'}_{-s}(\Omega)^n \\
& \Longrightarrow \vec{u} = 0 \ \text{ in } \ V^{-s,p'}(\Omega).
\end{aligned} \tag{7.30}$$

This shows that Φ in (7.27) is an isomorphism, thus finishing the proof of (7.25). The proof of the theorem is therefore completed. □

8. The Poisson problem for the Stokes operator with Neumann conditions

For a given Lipschitz domain Ω in $\mathbb{R}^n$, $n \geq 2$, the range of indices for which the Poisson problem in Ω for the Stokes operator equipped with Neumann boundary conditions is well-posed on Besov and Triebel-Lizorkin spaces depends on the dimension n of the ambient space and the Lipschitz character of Ω. The latter is manifested by a parameter $\varepsilon \in (0,1]$ which can be thought of as measuring the degree of roughness of Ω (thus, the larger ε the milder the Lipschitz nature of Ω, and the smaller ε, the more acute Lipschitz nature of Ω). To best describe these regions, for each $n \geq 2$ and $\varepsilon > 0$ we let $\mathcal{R}_{n,\varepsilon}$ denote the following sets. For $n = 2$, $\mathcal{R}_{2,\varepsilon}$ is the collection of all pairs of numbers s, p with the property that either one of the following two conditions below is satisfied:

$$\begin{aligned}
(I_2): &\quad 0 \leq \tfrac{1}{p} < s + \tfrac{1+\varepsilon}{2} \quad \text{and} \quad 0 < s \leq \tfrac{1+\varepsilon}{2}, \\
(II_2): &\quad -\tfrac{1+\varepsilon}{2} < \tfrac{1}{p} - s < \tfrac{1+\varepsilon}{2} \quad \text{and} \quad \tfrac{1+\varepsilon}{2} < s < 1.
\end{aligned} \tag{8.1}$$

Corresponding to $n = 3$, $\mathcal{R}_{3,\varepsilon}$ is the collection of all pairs s, p with the property that either of the following two conditions holds:

$$\begin{aligned}
(I_3): &\quad 0 \leq \tfrac{1}{p} < \tfrac{s}{2} + \tfrac{1+\varepsilon}{2} \quad \text{and} \quad 0 < s < \varepsilon, \\
(II_3): &\quad -\tfrac{\varepsilon}{2} < \tfrac{1}{p} - \tfrac{s}{2} < \tfrac{1+\varepsilon}{2} \quad \text{and} \quad \varepsilon \leq s < 1.
\end{aligned} \tag{8.2}$$

Finally, corresponding to $n \geq 4$, we let $\mathcal{R}_{n,\varepsilon}$ denote the collection of all pairs s, p with the property that

$$(I_n): \quad \tfrac{n-3}{2(n-1)} - \varepsilon < \tfrac{1}{p} - \tfrac{s}{n-1} < \tfrac{1}{2} + \varepsilon \ \text{ and } \ 0 < s < 1,\ 1 < p < \infty. \tag{8.3}$$

The following well-posedness result has been recently established in [32].

Theorem 8.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$, with connected complement, and fix $\frac{n-1}{n} < p \leq \infty$, $0 < q \leq \infty$, and $(n-1)\big(\frac{1}{p} - 1\big)_+ < s < 1$. Also,* *assume that* $\lambda \in (-1,1]$ *and* $\mu \in \mathbb{C} \setminus \mathrm{Spec}\,(B_\lambda)$. *Then there exists* $\varepsilon = \varepsilon(\Omega) \in (0,1]$ *such that the Poisson problem for the Stokes system with Neumann boundary condition*

$$(8.4)\qquad \begin{array}{l} \mu\,\vec{u} - \Delta\vec{u} + \nabla\pi = \vec{f}\Big|_{\Omega}, \quad \vec{f} \in B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)^n, \quad \mathrm{div}\,\vec{u} = 0 \text{ in } \Omega, \\ \vec{u} \in B^{p,q}_{s+\frac{1}{p}}(\Omega)^n, \quad \pi \in B^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}-\mu\,\vec{u}} = 0 \text{ in } B^{p,q}_{s-1}(\partial\Omega)^n, \end{array}$$

has a unique solution if the pair s,p *belongs to the region* $\mathcal{R}_{n,\varepsilon}$*, described in* (8.1)-(8.3). *In addition, the solution satisfies the estimate*

$$(8.5)\qquad \|\vec{u}\|_{B^{p,q}_{s+\frac{1}{p}}(\Omega)^n} + \|\pi\|_{B^{p,q}_{s+\frac{1}{p}-1}(\Omega)} \leq C\|\vec{f}\|_{B^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)^n},$$

for some finite constant $C = C(\Omega,n,p,s,\lambda,\mu) > 0$.

Moreover, an analogous well-posedness result holds for the problem

$$(8.6)\qquad \begin{array}{l} \mu\,\vec{u} - \Delta\vec{u} + \nabla\pi = \vec{f}\Big|_{\Omega}, \quad \vec{f} \in F^{p,q}_{s+\frac{1}{p}-2,0}(\Omega)^n, \quad \mathrm{div}\,\vec{u} = 0 \text{ in } \Omega, \\ \vec{u} \in F^{p,q}_{s+\frac{1}{p}}(\Omega)^n, \quad \pi \in F^{p,q}_{s+\frac{1}{p}-1}(\Omega), \quad \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}-\mu\,\vec{u}} = 0 \text{ in } B^{p,p}_{s-1}(\partial\Omega)^n, \end{array}$$

assuming that $p,q < \infty$.

Strictly speaking, the above theorem has been proved in [32] when $\mu = 0$ (in which case the data must satisfy certain necessary compatibility conditions, and uniqueness is valid up a finite dimensional space). The method of proof in [32] is constructive as it relies on integral representation formulas (involving hydrostatic potential operators). As such, this approach can be easily adapted to the slightly more general case above, since the difference between the fundamental solutions for the original Stokes system $\Big\{-\Delta\,\vec{u}+\nabla\pi = 0\,,\ \mathrm{div}\,\vec{u} = 0\Big\}$ and the lower-order perturbation $\Big\{(\mu-\Delta)\,\vec{u}+\nabla\pi = 0\,,\ \mathrm{div}\,\vec{u} = 0\Big\}$ is only weakly singular. We leave the straightforward details to the interested reader.

9. Domains of fractional powers of the Neumann Stokes operator: I

Here we study the global regularity, measured on the Sobolev scale, of vector fields in the domains of fractional powers of the Neumann Stokes operator. Our first result in this regard reads as follows:

Theorem 9.1. *Let* Ω *be a Lipschitz domain in* $\mathbb{R}^n$ *and fix* $\lambda \in (-1,1]$. *Then the domain of the fractional power of the Neumann Stokes operator* B_λ *introduced in* (6.2)-(6.3) *satisfies*

$$(9.1)\qquad D(B_\lambda^{\frac{s}{2}}) = \Big\{\vec{u} \in L^2_s(\Omega)^n :\ div\,\vec{u} = 0\Big\} \quad \text{if } \ 0 \leq s \leq 1,$$

and

$$(9.2)\qquad \vec{u} \in D(B_\lambda^{\frac{s}{2}}) \Longleftrightarrow \left\{ \begin{array}{l} \vec{u} \in \mathcal{V} \text{ and there exists } \pi \in L^2(\Omega) \text{ such that} \\ \vec{f} := -\Delta\,\vec{u} + \nabla\pi \in L^2_{s-2}(\Omega)^n \text{ and } \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}} = 0, \end{array} \right.$$

granted that $s \in (3/2, 2]$.

Proof. Consider the families of spaces $\Big\{V^{s,2}(\Omega)\,:\ s \in \mathbb{R}\Big\}$ and $\Big\{D(B^{\frac{s}{2}})\,:\ 0 \leq s \leq 2\Big\}$. From Theorem 7.1 and (5.11) we know that both are complex interpolation scales, and

$$(9.3)\qquad D(B^0_\lambda) = \mathcal{H} = V^{0,2}(\Omega), \qquad D(B_\lambda^{\frac{1}{2}}) = \mathcal{V} = V^{1,2}(\Omega).$$

Thus, by complex interpolation,

$$D(B_\lambda^{\frac{s}{2}}) = V^{s,2}(\Omega), \qquad 0 \le s \le 1, \tag{9.4}$$

which gives the description of $D(B_\lambda^{\frac{s}{2}})$ in (9.1).

To study larger values of s, recall the form $b_\lambda(\cdot,\cdot)$ and the operator $\widetilde{B}_\lambda$ induced by it. From (5.20)-(9.4) we obtain

$$D(B_\lambda^{\frac{s}{2}}) = (\widetilde{I} + \widetilde{B}_\lambda)^{-1}\Big(V^{2-s,2}(\Omega)\Big)^*, \qquad 1 \le s \le 2. \tag{9.5}$$

Thus, by (7.25),

$$\vec{u} \in D(B_\lambda^{\frac{s}{2}}) \Longleftrightarrow \vec{u} \in \mathcal{V} \text{ and } (\widetilde{I} + \widetilde{B}_\lambda)\vec{u} \in V^{s-2,2}(\Omega), \qquad \text{if } \tfrac{3}{2} < s \le 2. \tag{9.6}$$

Consequently, if $s \in (3/2, 2]$, then by taking into account the very definition of $\widetilde{B}_\lambda$ we arrive at the conclusion that

$$\vec{u} \in D(B_\lambda^{\frac{s}{2}}) \Longleftrightarrow \begin{cases} \vec{u} \in \mathcal{V} \text{ and } \exists\, \vec{f} \in L^2_{s-2}(\Omega)^n \text{ such that} \\ \langle \vec{f}, \vec{v}\rangle = \int_\Omega \overline{\vec{u}} \cdot \vec{v}\, dx + \int_\Omega A_\lambda(\overline{\nabla \vec{u}}, \nabla \vec{v})\, dx, \ \forall\, \vec{v} \in \mathcal{V}. \end{cases} \tag{9.7}$$

Much as before, by relying on Lemma 6.7, Corollary 3.3 and Proposition 3.6, it follows from (9.7) that

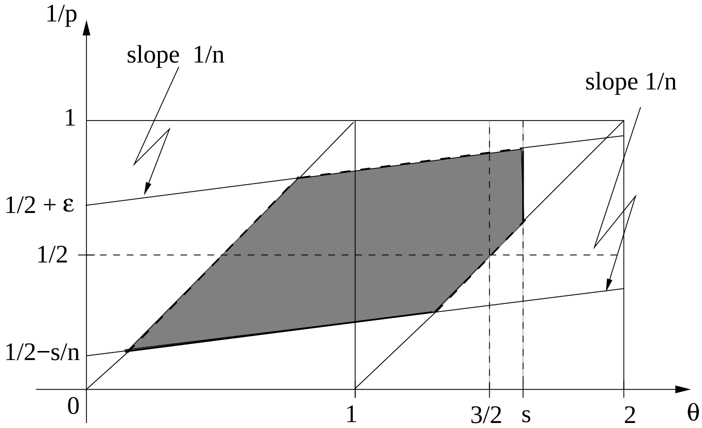
$$\vec{u} \in D(B_\lambda^{\frac{s}{2}}) \Longleftrightarrow \begin{cases} \vec{u} \in \mathcal{V} \text{ and there exists } \pi \in L^2(\Omega) \text{ such that} \\ \vec{f} := (1-\Delta)\,\vec{u} + \nabla\pi \in L^2_{s-2}(\Omega)^n \text{ and } \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}-\vec{u}} = 0, \end{cases} \tag{9.8}$$

With this in hand, (9.2) follows after re-denoting $\vec{f} - \vec{u}$ by $\vec{f}$. □

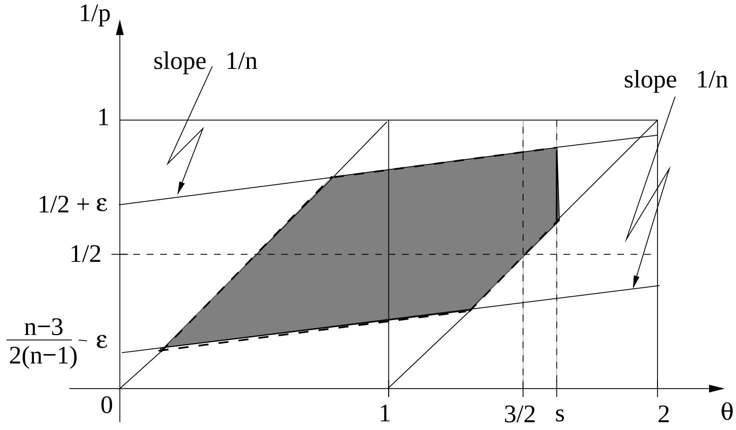
It is possible to further extend the scope of the above analysis. In order to facilitate the subsequent discussion, for each $\varepsilon \in (0,1]$, $s \in [\frac{3}{2}, 2]$ and $n \ge 2$, define the two dimensional region

$$R_{n,s,\varepsilon} := \begin{cases} (\theta, \frac{1}{p}) : \ 0 < \frac{1}{p} < \theta < 1 + \frac{1}{p} < 2, \ \theta \le s, \quad \text{and} \\ \frac{1}{2} + \varepsilon > \frac{1}{p} - \frac{\theta}{n} \ge \frac{1}{2} - \frac{s}{n} \quad \text{if} \quad \frac{3}{2} \le s < \frac{n}{n-1} + \varepsilon n, \\ \frac{1}{2} + \varepsilon > \frac{1}{p} - \frac{\theta}{n} > -\frac{\varepsilon}{n} \quad \text{if} \quad \frac{n}{n-1} + \varepsilon n < s \le 2. \end{cases} \tag{9.9}$$

The figures below depict the region $R_{n,s,\varepsilon}$ in the case when $\frac{3}{2} \le s < \frac{n}{n-1} + \varepsilon n$,



and when $\frac{n}{n-1} + \varepsilon n < s \le 2$, respectively:



Theorem 9.2. *For every Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, there exists $\varepsilon = \varepsilon(\partial\Omega) > 0$ with the property that for every $s \in (3/2, 2]$ and $\lambda \in (-1, 1]$ the following implication holds:*

$$(\theta, 1/p) \in R_{n,s,\varepsilon} \Longrightarrow D(B_\lambda^{s/2}) \subset L_\theta^p(\Omega)^n. \tag{9.10}$$

Proof. The strategy is to combine the characterization (9.2) with the well-posedness result for the Poisson problem for the Stokes system equipped with Neumann boundary conditions. In concert, these two results show that $D(B_\lambda^{\alpha/2}) \subset L_\theta^p(\Omega)^n$ provided

$$\begin{aligned} &\exists\, s, p \text{ belonging to the region } \mathcal{R}_{n,\varepsilon} \text{ such that} \\ &\theta = s + 1/p \quad \text{and} \quad L_{\alpha-2}^2(\Omega) \hookrightarrow L_{\theta-2}^p(\Omega). \end{aligned} \tag{9.11}$$

Now, elementary algebra shows that, given $\alpha \in (3/2, 2]$, the condition (9.11) holds if and only if $(\theta, 1/p) \in R_{n,\alpha,\varepsilon}$. Clearly, this proves (9.10), after re-adjusting notation. ☐

Corollary 9.3. *For a Lipschitz domain Ω in $\mathbb{R}^n$ one has*

$$D(B_\lambda^\alpha) \subset \bigcup_{p > \frac{2n}{n-1}} L_1^p(\Omega)^n \quad \text{if} \quad \alpha > \tfrac{3}{4}. \tag{9.12}$$

Also, when $n = 3$,

$$D(B_\lambda^\alpha) \subset \mathscr{C}^{2\alpha-3/2}(\bar{\Omega})^3 \quad \text{if} \quad \tfrac{3}{4} < \alpha < \tfrac{3}{4} + \varepsilon, \tag{9.13}$$

and when $n = 2$,

$$D(B_\lambda^\alpha) \subset \mathscr{C}^{2\alpha-1}(\bar{\Omega})^2 \quad \text{if} \quad \tfrac{3}{4} < \alpha < \tfrac{3}{4} + \varepsilon, \tag{9.14}$$

for some small $\varepsilon = \varepsilon(\Omega) > 0$.

Proof. These are all immediate consequences of Theorem 9.2 and classical embedding results. ☐

10. Domains of fractional powers of the Neumann Stokes operator: II

The aim of this section is to augment the results in Theorem 9.1 by including a description of $D(B_\lambda^{s/2})$ in the case when $s \in (1, 3/2]$. See Theorem 10.5 below. We begin by revisiting the Neumann-Leray projection (7.22), with the goal of further extending the range of action of this operator.

Lemma 10.1. *Assume that Ω is a Lipschitz domain in $\mathbb{R}^n$ and that $s \in \mathbb{R}$, $p, p' \in (1, \infty)$, $1/p + 1/p' = 1$. Then the operator*

$$\widehat{\mathbb{P}}_{s,p} : L_{-s,0}^{p'}(\Omega)^n = \big(L_s^p(\Omega)^n\big)^* \longrightarrow \big(V^{s,p}(\Omega)\big)^* \tag{10.1}$$

(where we have used the isomorphism Ψ *defined in* (3.15) *to identify* $L^{p'}_{-s,0}(\Omega)^n$ *and* $\big(L^p_s(\Omega)^n\big)^*$*) defined by the requirement that*

$$(10.2)\qquad {}_{V^{s,p}(\Omega)}\big\langle \vec{v}, \widehat{\mathbb{P}}_{s,p}\vec{u}\big\rangle_{(V^{s,p}(\Omega))^*} = {}_{L^p_s(\Omega)^n}\big\langle J_{s,p}\vec{v}, \vec{u}\big\rangle_{(L^p_s(\Omega)^n)^*} \qquad \forall\,\vec{v}\in V^{s,p}(\Omega),$$

is well-defined, linear, bounded and onto. In addition, any two such operators act coherently, i.e., $\widehat{\mathbb{P}}_{s_1,p_1} = \widehat{\mathbb{P}}_{s_2,p_2}$ *on* $L^{p'_1}_{-s_1,0}(\Omega)^n \cap L^{p'_2}_{-s_2,0}(\Omega)^n$ *for any numbers* $s_1, s_2 \in \mathbb{R}$ *and* $p_1, p_2 \in (1,\infty)$*. Also, the null-space of* (10.1)-(10.2) *is*

$$(10.3)\qquad \mathrm{Ker}\Big[\widehat{\mathbb{P}} : L^{p'}_{-s,0}(\Omega)^n \longrightarrow \Big(V^{s,p}(\Omega)\Big)^*\Big] = \nabla\Big[L^{p'}_{1-s,0}(\Omega)\Big].$$

Finally, if corresponding to $s=1$ *and* $p=2$ *one considers*

$$(10.4)\qquad \begin{aligned} &\widehat{\mathbb{P}}_{1,2} : L^2_{-1,0}(\Omega)^n = \big(L^2_1(\Omega)^n\big)^* \longrightarrow \mathcal{V}^*,\\ &{}_{\mathcal{V}}\big\langle \vec{v}, \widehat{\mathbb{P}}_{1,2}\vec{u}\big\rangle_{\mathcal{V}^*} = {}_{L^2_1(\Omega)^n}\big\langle J_{1,2}\vec{v}, \vec{u}\big\rangle_{(L^2_1(\Omega)^n)^*} \qquad \forall\,\vec{v}\in\mathcal{V}, \end{aligned}$$

then the diagram

$$(10.5)\qquad \begin{array}{ccc} L^2_{-1,0}(\Omega)^n & \xrightarrow{\ \widehat{\mathbb{P}}_{1,2}\ } & \mathcal{V}^* \\ \Big\uparrow & & \Big\uparrow \\ L^2(\Omega)^n & \xrightarrow{\ \mathbb{P}\ } & \mathcal{H} \end{array}$$

in which the vertical arrows are natural inclusions, is commutative. Consequently, the Neumann-Leray projection (7.22) *extends as in* (10.4)*.*

Proof. That (10.1)-(10.2) is well-defined and bounded is clear from the continuity of the inclusion $J_{s,p} : V^{s,p}(\Omega) \hookrightarrow L^p_s(\Omega)^n$. Using the fact that $V^{s,p}(\Omega)$ is a closed subspace of $L^p_s(\Omega)^n$, the Hahn-Banach theorem, and (3.15), it is straightforward to show that the operator (10.1) is onto. It is also clear from (10.2) that this family of operators acts in a mutually compatible fashion.

The left-to-right inclusion in (10.3) is a direct consequence of the fact that

$$(10.6)\qquad {}_{L^p_s(\Omega)^n}\Big\langle J_{s,p}\vec{v}\,,\,\nabla\pi\Big\rangle_{L^{p'}_{-s,0}(\Omega)^n} = 0, \qquad \forall\,\vec{v}\in V^{s,p}(\Omega), \quad \forall\,\pi\in L^{p'}_{1-s,0}(\Omega).$$

In turn, (10.6) follows from a standard density argument (based on the fact that the inclusion (3.16) has dense range), and the fact that vector fields in $V^{s,p}(\Omega)$ are, in the sense of distribution, divergence-free. To prove the opposite inclusion in (10.3), assume that $\vec{u}\in L^{p'}_{-s,0}(\Omega)^n$ is such that

$$(10.7)\qquad {}_{L^p_s(\Omega)^n}\big\langle J_{s,p}\vec{v}, \vec{u}\big\rangle_{(L^p_s(\Omega)^n)^*} = 0, \qquad \forall\,\vec{v}\in V^{s,p}(\Omega).$$

Then, on account of (3.18), for every $\vec{w}\in C^\infty_c(\mathbb{R}^n)^n$ such that $\mathrm{div}\,\vec{w} = 0$ in $\mathbb{R}^n$ we have

$$(10.8)\qquad {}_{L^p_s(\mathbb{R}^n)^n}\big\langle \vec{w}, \vec{u}\big\rangle_{L^{p'}_{-s}(\mathbb{R}^n)^n} = {}_{L^p_s(\Omega)^n}\big\langle \mathscr{R}_\Omega\vec{w}, \vec{u}\big\rangle_{L^{p'}_{-s,0}(\Omega)^n} = 0,$$

thanks to (10.7), used with $\vec{v} := \mathscr{R}_\Omega\vec{w}$. With this in hand, Lemma 6.7 then shows that $\vec{u} = \nabla\pi$ for some distribution π in $\mathbb{R}^n$. In fact, since $\vec{u}$ is supported in $\overline{\Omega}$ and $\mathbb{R}^n\setminus\Omega$ is connected, after eventually subtracting a constant it can be arranged that π is also supported in $\overline{\Omega}$. Finally, Proposition 3.2 gives that $\pi\in L^{p'}_{1-s}(\mathbb{R}^n)$ so that, all together, $\pi\in L^{p'}_{1-s,0}(\Omega)$. This shows that $\vec{u}\in\nabla\Big[L^{p'}_{1-s,0}(\Omega)\Big]$ and completes the proof of the right-to-left inclusion in (10.3). Thus, (10.3) holds.

Next, to show that the diagram (10.5) is commutative, pick $\vec{u} \in L^2(\Omega)^n$ and use (7.21) (with $s = 0$ and $p = 2$) in order to decompose it as $\mathbb{P}\vec{u} + \nabla\pi$ for some $\pi \in L^2_{1,z}(\Omega)$. Then, since by (10.6)

$$(J_{1,2}\vec{v}, \nabla\pi)_{L^2(\Omega)^n} = 0 \qquad \forall\, \vec{v} \in \mathcal{V}, \tag{10.9}$$

for every $\vec{v} \in \mathcal{V}$ we have

$$\begin{aligned} {}_{\mathcal{V}}\big\langle J_{1,2}\vec{v}, \widehat{\mathbb{P}}_{1,2}\vec{u}\big\rangle_{\mathcal{V}^*} &= {}_{L^2_1(\Omega)^n}\big\langle J_{1,2}\vec{v}, \vec{u}\big\rangle_{(L^2_1(\Omega)^n)^*} = \big(J_{1,2}\vec{v}, \vec{u}\big)_{L^2(\Omega)^n} \\ &= \big\langle J_{1,2}\vec{v}, \mathbb{P}\vec{u}\big\rangle_{L^2(\Omega)^n} + \big(J_{1,2}\vec{v}, \nabla\pi\big)_{L^2(\Omega)^n} = {}_{\mathcal{V}}\big\langle \vec{v}, \mathbb{P}\vec{u}\big\rangle_{\mathcal{V}^*}. \end{aligned} \tag{10.10}$$

This shows that $\widehat{\mathbb{P}}_{1,2}\vec{u} = \mathbb{P}\vec{u}$ in $\mathcal{V}^*$, as desired. $\square$

Remark 10.2. A close inspection of the above argument shows that, for each $p \in (1,\infty)$ and $s \in \mathbb{R}$, the operator (10.1)-(10.2) factors as

$$\widehat{\mathbb{P}} : L^{p'}_{-s,0}(\Omega)^n \longrightarrow \frac{L^{p'}_{-s,0}(\Omega)^n}{\nabla\big[L^{p'}_{1-s,0}(\Omega)\big]} \longrightarrow \Big(V^{s,p}(\Omega)\Big)^*, \tag{10.11}$$

where the first arrow is the canonical projection onto the quotient space, and the second arrow is an isomorphism naturally induced by $J^*_{s,p}$, the adjoint of (7.15).

Next, let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and assume that $\lambda \in (-1,1]$ has been fixed. Recall the operator $\widetilde{B}_\lambda$ induced by the sesquilinear form $b_\lambda(\cdot,\cdot)$, i.e.,

$$\widetilde{B}_\lambda : \mathcal{V} \longrightarrow \mathcal{V}^*, \qquad \widetilde{B}_\lambda\vec{u} := b_\lambda(\cdot,\vec{u}) \in \mathcal{V}^*, \quad \vec{u} \in \mathcal{V}. \tag{10.12}$$

Next, fix $\vec{u} \in V$, so that $\widetilde{B}_\lambda\vec{u} : \mathcal{V} \to \mathbb{C}$ is a linear, bounded functional. Since $\mathcal{V}$ is a closed subspace of $L^2_1(\Omega)^n$, the Hahn-Banach theorem ensures the existence of some linear, bounded functional $\vec{f} : L^2_1(\Omega)^n \to \mathbb{C}$ with the property that $\vec{f}|_{\mathcal{V}} = (\widetilde{I} + \widetilde{B}_\lambda)\vec{u}$. Thus, $\vec{f} \in \big(L^2_1(\Omega)^n\big)^* = L^2_{1,0}(\Omega)^n$ satisfies

$$\begin{aligned} {}_{L^2_1(\Omega)^n}\big\langle \vec{v}, \vec{f}\big\rangle_{(L^2_1(\Omega)^n)^*} &= {}_{\mathcal{V}}\big\langle \vec{v}, (\widetilde{I} + \widetilde{B}_\lambda)\vec{u}\big\rangle_{\mathcal{V}^*} \\ &= \int_\Omega \overline{\vec{u}} \cdot \vec{v}\, dx + \int_\Omega A_\lambda(\overline{\nabla\vec{u}}, \nabla\vec{v})\, dx, \qquad \vec{v} \in \mathcal{V} \hookrightarrow L^2_1(\Omega)^n. \end{aligned} \tag{10.13}$$

Specializing this to the case when $\vec{v}$ belongs to $\{\vec{v} \in \mathscr{C}^\infty_c(\Omega)^n : \operatorname{div}\vec{v} = 0 \text{ in } \Omega\}$ shows that the distribution $\vec{f}|_\Omega - (1-\Delta)\,\vec{u} \in L^2_{-1}(\Omega)^n$ annihilates this space. Thus, by Lemma 6.7, there exists a distribution π in Ω such that

$$\nabla\pi = \vec{f}\Big|_\Omega - (1-\Delta)\vec{u} \in L^2_{-1}(\Omega)^n. \tag{10.14}$$

In particular, $\pi \in L^2(\Omega)$ by Corollary 3.3. Returning with this information back in (10.13) and invoking (3.38) then shows that, after an eventual re-normalization of π (done by subtracting a suitable constant, similar in spirit to (6.23)), matters can be arranged so that

$$\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}-\vec{u}} = 0 \quad \text{in} \quad L^2_{-1/2}(\partial\Omega)^n. \tag{10.15}$$

The stage is now set for proving the following result.

Proposition 10.3. *Suppose that $\Omega \subset \mathbb{R}^n$ is a Lipschitz domain and that $\lambda \in (-1,1]$. Then for every $\vec{u} \in \mathcal{V}$ there exist*

$$\pi \in L^2(\Omega) \quad \textit{and} \quad \vec{f} \in L^2_{-1,0}(\Omega)^n \tag{10.16}$$

such that

$$(1-\Delta\,)\vec{u}+\nabla\pi=\vec{f}\Big|_{\Omega}\quad in\quad L^2_{-1}(\Omega)^n, \tag{10.17}$$

$$\partial^{\lambda}_{\nu}(\vec{u},\pi)_{\vec{f}-\vec{u}}=0\quad in\quad L^2_{-1/2}(\partial\Omega)^n, \tag{10.18}$$

$$and\quad (\widetilde{I}+\widetilde{B}_{\lambda})\vec{u}=\widehat{\mathbb{P}}_{1,2}\vec{f}\quad in\quad \mathcal{V}^*. \tag{10.19}$$

Furthermore, if $\vec{g}\in L^2_{-1,0}(\Omega)^n$ *is such that* $\widehat{\mathbb{P}}_{1,2}\,\vec{g}=\widehat{\mathbb{P}}_{1,2}\,\vec{f}$, *then there exists* $q\in L^2(\Omega)$ *with the property that*

$$(1-\Delta\,)\vec{u}+\nabla(\pi-q)=\vec{g}\Big|_{\Omega}\quad in\quad L^2_{-1}(\Omega)^n, \tag{10.20}$$

$$\partial^{\lambda}_{\nu}(\vec{u},\pi-q)_{\vec{g}-\vec{u}}=0\quad in\quad L^2_{-1/2}(\partial\Omega)^n. \tag{10.21}$$

Proof. The existence of π, $\vec{f}$ as in (10.16) and for which (10.17)-(10.18) are satisfied is clear from the discussion preceding the statement of the proposition. Hence, there remains to prove (10.19). This, however, is a direct consequence of Lemma 10.1 and the first equality in (10.13).

There remains to take care of the claim in the second part of the statement. To this end, we first note that $\widehat{\mathbb{P}}_{1,2}(\vec{f}-\vec{g})=0$ entails

$${}_{L^2_1(\Omega)^n}\big\langle J_{1,2}\vec{v},\vec{f}-\vec{g}\big\rangle_{(L^2_1(\Omega)^n)^*}=0,\qquad \forall\,\vec{v}\in\mathcal{V}. \tag{10.22}$$

Thus, based on (10.3) we may conclude that there exists some scalar function $\hat{q}\in L^2(\Omega)$ with the property that $(\vec{f}-\vec{g})|_{\Omega}=\nabla\hat{q}$ in $L^2_{-1}(\Omega)$. In turn, this and (10.17) yield

$$(1-\Delta\,)\vec{u}+\nabla(\pi-\hat{q})=\vec{g}\Big|_{\Omega}\quad \text{in}\quad L^2_{-1}(\Omega)^n. \tag{10.23}$$

Going further, formula (3.38) gives that for every $\vec{w}\in\mathcal{V}\hookrightarrow L^2_1(\Omega)^n$

$$\begin{aligned}\Big\langle \mathrm{Tr}\,\vec{w}\,,\,\partial^{\lambda}_{\nu}(\vec{u},\pi-\hat{q})_{\vec{g}-\vec{u}}\Big\rangle &= \int_{\Omega}\overline{\vec{w}}\cdot\vec{u}\,dx+\mathbb{A}_{\lambda}\Big(\overline{\nabla\vec{w}},\nabla\vec{u}\Big)\\ &\quad -{}_{L^2_1(\Omega)^n}\big\langle J_{1,2}\vec{w},\vec{g}\big\rangle_{(L^2_1(\Omega)^n)^*}.\end{aligned} \tag{10.24}$$

On the other hand, for every $\vec{w}\in\mathcal{V}$ we have

$$\begin{aligned}{}_{L^2_1(\Omega)^n}\big\langle J_{1,2}\vec{w},\vec{g}\big\rangle_{(L^2_1(\Omega)^n)^*} &= {}_{\mathcal{V}}\langle\vec{w},\widehat{\mathbb{P}}_{1,2}\,\vec{g}\rangle_{\mathcal{V}^*}={}_{\mathcal{V}}\langle\vec{w},\widehat{\mathbb{P}}_{1,2}\,\vec{f}\rangle_{\mathcal{V}^*}\\ &= {}_{L^2_1(\Omega)^n}\big\langle J_{1,2}\vec{w},\vec{f}\big\rangle_{(L^2_1(\Omega)^n)^*}\\ &= \int_{\Omega}\overline{\vec{w}}\cdot\vec{u}\,dx+\mathbb{A}_{\lambda}\Big(\overline{\nabla\vec{w}},\nabla\vec{u}\Big),\end{aligned} \tag{10.25}$$

by hypotheses, (10.4), (3.38) and (10.18). Together, this and (10.24) then prove that

$$\Big\langle\partial^{\lambda}_{\nu}(\vec{u},\pi-\hat{q})_{\vec{g}-\vec{u}}\,,\,\mathrm{Tr}\,\vec{w}\Big\rangle=0,\qquad\forall\,\vec{w}\in\mathcal{V}\hookrightarrow L^2_1(\Omega)^n. \tag{10.26}$$

With this in hand, and by proceeding as in (6.24)-(6.29), we may then conclude that there exists a constant $c\in\mathbb{R}$ with the property that if $q:=\hat{q}-c$ then (10.20)-(10.21) hold. $\square$

Once again, suppose that $\Omega\subset\mathbb{R}^n$ is a Lipschitz domain and that $\lambda\in(-1,1]$. Also, fix $p\in(1,\infty)$ and assume that $1/p<s<1+1/p$, $1<p'<\infty$, $1/p+1/p'=1$. Then the operator $\widetilde{B}_\lambda$ from (10.12) extends to a bounded mapping

$$\begin{aligned}&\widetilde{B}_\lambda : V^{s,p}(\Omega) \longrightarrow \Big(V^{2-s,p'}(\Omega)\Big)^*,\\ &\widetilde{B}_\lambda\vec{u} := \mathbb{A}_\lambda(\cdot,\vec{u}) \in \Big(V^{2-s,p'}(\Omega)\Big)^*, \quad \vec{u}\in V^{s,p}(\Omega).\end{aligned} \tag{10.27}$$

A similar line of reasoning as in the proof of Proposition 10.3 (the only significant difference is that Proposition 3.2 is used in place of Corollary 3.3) then yields the following.

Proposition 10.4. *Retain the above notation and conventions. Also, assume that $\mu\in\mathbb{R}$. Then for every $\vec{u}\in V^{s,p}(\Omega)$ there exist*

$$\pi\in L^p_{s-1}(\Omega) \quad \textit{and} \quad \vec{f}\in L^P_{s+1/p-2,0}(\Omega)^n \tag{10.28}$$

such that

$$(\mu-\Delta\,)\vec{u}+\nabla\pi = \vec{f}\Big|_\Omega \quad \textit{in} \;\; L^p_{s+1/p-2}(\Omega)^n, \tag{10.29}$$

$$\partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}-\mu\,\vec{u}} = 0 \quad \textit{in} \;\; B^{p,p}_{s-1}(\partial\Omega)^n, \tag{10.30}$$

$$\textit{and} \;\; (\mu\widetilde{I}+\widetilde{B}_\lambda)\vec{u} = \widehat{\mathbb{P}}_{s,p}\vec{f} \quad \textit{in} \;\; \Big(V^{2-s,p'}(\Omega)\Big)^*. \tag{10.31}$$

The stage has now been set for us to prove the following.

Theorem 10.5. *Let $\Omega\subset\mathbb{R}^n$ be a Lipschitz domain and assume that $\lambda\in(-1,1]$. Then the domain of the fractional power of the Neumann-Stokes operator B_λ satisfies*

$$D(B_\lambda^{\frac{s}{2}}) = \Big\{\vec{u}\in L^2_s(\Omega)^n : \, div\,\vec{u}=0 \;\; in\;\Omega\Big\} \quad if \quad s\in(1,\tfrac{3}{2}). \tag{10.32}$$

Furthermore, corresponding to $s=3/2$, one has that $\vec{u}\in D(B_\lambda^{\frac{3}{4}})$ if and only if

$$\begin{cases} \vec{u}\in\mathcal{V} \;\textit{and}\; \exists\,\pi\in L^2(\Omega),\;\; \exists\,\vec{f}\in L^2_{-1/2,0}(\Omega)^n \hookrightarrow L^2_{-1,0}(\Omega)^n,\\ \textit{such that}\; (1-\Delta)\,\vec{u}+\nabla\pi = \vec{f}\Big|_\Omega \quad \textit{in} \;\; L^2_{-1/2}(\Omega)^n \hookrightarrow L^2_{-1}(\Omega)^n,\\ \textit{and for which} \;\; \partial^\lambda_\nu(\vec{u},\pi)_{\vec{f}-\vec{u}} = 0 \quad \textit{in} \;\; L^2_{-1/2}(\partial\Omega)^n. \end{cases} \tag{10.33}$$

Proof. Assume that $s\in[1,2]$ and recall (9.5). Much as with (9.6), we have

$$\vec{u}\in D(B_\lambda^{\frac{s}{2}}) \Longleftrightarrow \vec{u}\in\mathcal{V} \text{ and } (\widetilde{I}+\widetilde{B}_\lambda)\vec{u}\in\Big(V^{2-s,2}(\Omega)\Big)^* \hookrightarrow \mathcal{V}^*. \tag{10.34}$$

Now, given $\vec{u}\in D(B_\lambda^{\frac{s}{2}})$, Proposition 10.3 ensures that there exist $\vec{f}$, π as in (10.16) such that (10.17)-(10.19) are satisfied. On the other hand, from Lemma 10.1 we know that the operator (10.1) is onto. This implies that there exists $\vec{g}\in L^2_{s-2,0}(\Omega)^n$ such that $\widehat{\mathbb{P}}_{1,2}\,\vec{g} = (\widetilde{I}+\widetilde{B}_\lambda)\vec{u}$ in $\mathcal{V}^*$. Then, according to the second part in the statement of Proposition 10.3, there exists $q\in L^2(\Omega)$ such that (10.20)-(10.21) hold. As a consequence, if $\widetilde{\pi} := \pi - q$, then for each $s\in[1,2]$,

$$\vec{u}\in D(B_\lambda^{\frac{s}{2}}) \Longleftrightarrow \begin{cases} \vec{u}\in\mathcal{V} \text{ and } \exists\,\widetilde{\pi}\in L^2(\Omega),\;\; \exists\,\vec{g}\in L^2_{s-2,0}(\Omega)^n \hookrightarrow L^2_{-1,0}(\Omega)^n,\\ \text{such that } (1-\Delta)\,\vec{u}+\nabla\widetilde{\pi} = \vec{g}\Big|_\Omega \;\text{ in }\; L^2_{s-2}(\Omega)^n \hookrightarrow L^2_{-1}(\Omega)^n,\\ \text{and for which } \;\partial^\lambda_\nu(\vec{u},\widetilde{\pi})_{\vec{g}-\vec{u}} = 0 \;\text{ in }\; L^2_{-1/2}(\partial\Omega)^n. \end{cases} \tag{10.35}$$

After adjusting notation, this equivalence with $s=3/2$ proves (10.33).

Assume next that $s \in (1, \frac{3}{2})$. With $\vec{u}$, $\widetilde{\pi}$ and $\vec{g}$ as in the right-hand side of (10.35), let $(\vec{w}, \rho)$ solve

$$\begin{cases} \vec{u} \in L^2_s(\Omega)^n, \quad \rho \in L^2_{s-1}(\Omega), \\ (1-\Delta)\,\vec{w} + \nabla\rho = \vec{g}\Big|_{\Omega}, \\ \operatorname{div}\vec{w} = 0 \quad \text{in} \quad \Omega, \\ \partial^\lambda_\nu(\vec{w},\rho)_{\vec{g}-\vec{w}} = 0 \quad \text{in} \quad L^2_{s-3/2}(\partial\Omega)^n. \end{cases} \tag{10.36}$$

That this is possible is ensured by Theorem 8.1. Then the difference $(\vec{v}, \eta) := (\vec{u}, \widetilde{\pi}) - (\vec{w}, \rho)$ solves the homogeneous system

$$\begin{cases} \vec{v} \in L^2_s(\Omega)^n, \quad \eta \in L^2_{s-1}(\Omega), \\ (1-\Delta)\,\vec{v} + \nabla\eta = 0 \quad \text{in} \quad \Omega, \\ \operatorname{div}\vec{v} = 0 \quad \text{in} \quad \Omega, \\ \partial^\lambda_\nu(\vec{v},\eta)_{-\vec{v}} = 0 \quad \text{in} \quad L^2_{-1/2}(\partial\Omega)^n. \end{cases} \tag{10.37}$$

This then forces $\partial^\lambda_\nu(\vec{v},\eta)_{-\vec{v}} = 0$ in $L^2_{s-3/2}(\partial\Omega)^n$ and, hence, $\vec{v} = 0$, $\eta = 0$ in Ω by the uniqueness part in Theorem 8.1. Thus, ultimately, $\vec{u} = \vec{w} \in L^2_s(\Omega)^n$ and $\widetilde{\pi} = \rho \in L^2_{s-1}(\Omega)$.

This proves the left-to-right inclusion in (10.32). The opposite implication in (10.32) then follows from (10.35) and Proposition 10.4 (considered with $p = 2$ and $\mu = 1$). □

Having established Theorem 10.5, the same argument as in the proof of Theorem 9.2 yields the following:

Corollary 10.6. *The end-point case $s = 3/2$ in (9.10) holds as well. As a corollary, if $n = 3$ then*

$$D(B_\lambda^{\frac{3}{4}}) \subset L^3_1(\Omega)^3. \tag{10.38}$$

It should be noted that this is the counterpart of a similar result for the Dirichlet-Stokes operator first established in [29].

We conclude this section with the following remark, whose veracity is apparent from a close inspection of earlier proofs:

Remark 10.7. All regularity results established in this paper for $D(B_\lambda^\alpha)$ are also valid in the case of $D((\mu I + B_\lambda)^\alpha)$ if $\mu \geq 0$. Furthermore, results similar in spirit hold in the case of the Stokes system with Neumann boundary conditions considered in Lipschitz subdomains of Riemannian manifolds (cf. [29] for the Dirichlet-Stokes operator in such a setting).

11. Navier-Stokes equations

In this section, we make use of our earlier analysis of the fractional powers of the Stokes system in order to study issues such as existence, uniqueness and regularity for the Navier-Stokes system in bounded Lipschitz subdomains of $\mathbb{R}^3$, in the sense of mild solutions as in (1.22). We are interested in the critical case, as far as functional spaces are concerned, i.e., $\vec{u}_0 \in D(B_\lambda^{1/4})$, where B_λ has been defined in Section 6. The ideas here follow the lines developed in [29] in the case of Dirichlet boundary conditions.

11.1. **Existence.** Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$ and fix $\lambda \in (-1,1]$. Then for each $T>0$, define the following Banach space:

$$\mathcal{F}_T := \Big\{\vec{u} \in \mathscr{C}([0,T];D(B_\lambda^{\frac14})) \cap \mathscr{C}^1((0,T];D(B_\lambda^{\frac34})) :$$

$$\sup_{0<s<T} \|s^{\frac12} B_\lambda^{\frac34}\vec{u}(s)\|_{\mathcal{H}} + \sup_{0<s<T} \|s^{\frac34}\vec{u}'(s)\|_{\mathcal{H}} + \sup_{0<s<T} \|s^{\frac32} B_\lambda^{\frac34}\vec{u}'(s)\|_{\mathcal{H}} < \infty\Big\}, \tag{11.1}$$

where $\mathcal{H}$ and B_λ are as in Section 6, endowed with the norm

$$\|\vec{u}\|_{\mathcal{F}_T} := \sup_{0<s<T} \|B_\lambda^{\frac14}\vec{u}(s)\|_{\mathcal{H}} + \sup_{0<s<T} \|s^{\frac12} B_\lambda^{\frac34}\vec{u}(s)\|_{\mathcal{H}}$$

$$+ \sup_{0<s<T} \|s^{\frac34}\vec{u}'(s)\|_{\mathcal{H}} + \sup_{0<s<T} \|s^{\frac32} B_\lambda^{\frac34}\vec{u}'(s)\|_{\mathcal{H}}. \tag{11.2}$$

Following (5.2)-(5.3), $-B_\lambda$ generates an analytic semigroup. For the convenience of notation, let us denote the Neumann-Stokes semigroup by

$$(S\vec{u}_0)(t) := e^{-tB_\lambda}\vec{u}_0, \qquad \vec{u}_0 \in \mathcal{H}, \quad t \geq 0, \tag{11.3}$$

Lemma 11.1. *If $\vec{u}_0 \in D(B_\lambda^{\frac14})$ then $S\vec{u}_0 \in \mathcal{F}_T$ for each $T>0$ and*

$$\|S\vec{u}_0\|_{\mathcal{F}_T} \leq C\|B_\lambda^{\frac14}\vec{u}_0\|_{\mathcal{H}} \tag{11.4}$$

where $C>0$ is a finite constant independent of $T>0$.

Proof. Fix some number $T>0$, as well as a divergence-free vector field $\vec{u}_0 \in D(B_\lambda^{\frac14})$. Since $(S\vec{u}_0)'(t) = -B_\lambda e^{-tB_\lambda}\vec{u}_0$ for $t>0$, it follows from (5.11) that

$$S\vec{u}_0 \in \mathscr{C}([0,T];D(B_\lambda^{\frac14})) \cap \mathscr{C}^1((0,T];D(B_\lambda^{\frac34})). \tag{11.5}$$

We also have that

$$t^{\frac12} B_\lambda^{\frac34}(S\vec{u}_0)(t) = t^{\frac12} B_\lambda^{\frac12} e^{-tB_\lambda} B_\lambda^{\frac14}\vec{u}_0 \tag{11.6}$$

is bounded from $(0,T)$ into $\mathcal{H}$. Likewise, the functions

$$t^{\frac32} B_\lambda^{\frac34}(S\vec{u}_0)'(t) = -t^{\frac32} B_\lambda^{\frac32} e^{-tB_\lambda} B_\lambda^{\frac14}\vec{u}_0 \tag{11.7}$$

and

$$t^{\frac34}(S\vec{u}_0)'(t) = -t^{\frac34} B_\lambda^{\frac34} e^{-tB_\lambda} B_\lambda^{\frac14}\vec{u}_0 \tag{11.8}$$

are bounded from $(0,T)$ into $\mathcal{H}$. This proves that $S\vec{u}_0 \in \mathcal{F}_T$. Now, (11.4) is implicit in the above analysis. □

Recall the operator $\mathbb{P}$ from (7.22) for $p=2$, $s \in (-\frac12,\frac12)$ and, for each $\vec{u},\vec{v} \in \mathcal{F}_T$, introduce

$$\Phi(\vec{u},\vec{v})(t) := \int_0^t e^{-(t-s)B_\lambda}(-\tfrac12\mathbb{P})((\vec{u}(s)\cdot\nabla)\vec{v}(s) + (\vec{v}(s)\cdot\nabla)\vec{u}(s))\,ds, \quad 0<t<T. \tag{11.9}$$

Proposition 11.2. *The application*

$$\Phi : \mathcal{F}_T \times \mathcal{F}_T \longrightarrow \mathcal{F}_T \tag{11.10}$$

is well-defined, bilinear, symmetric and continuous. Furthermore,

$$\|\Phi(\vec{u},\vec{v})\|_{\mathcal{F}_T} \leq \kappa\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T}, \qquad \vec{u},\vec{v} \in \mathcal{F}_T, \tag{11.11}$$

where $\kappa = \kappa(\Omega) > 0$ is a finite constant, independent of T.

Proof. The fact that Φ is bilinear and symmetric is clear. Moreover, $\Phi(\vec{u},\vec{v}) = e^{-\cdot B_\lambda} * \vec{f}$, where $\vec{f}$ is defined by

$$\vec{f}(s) := (-\tfrac{1}{2}\mathbb{P})((\vec{u}(s)\cdot\nabla)\vec{v}(s) + (\vec{v}(s)\cdot\nabla)\vec{u}(s)), \quad 0 < s < T. \tag{11.12}$$

We have $D(B_\lambda^{\frac{3}{4}}) \subset L^3_1(\Omega,\mathbb{R}^3)$ by (10.38) and $[D(B_\lambda^{\frac14}), D(B_\lambda^{\frac34})]_{\frac12} = D(B_\lambda^{\frac12}) \subset L^6(\Omega,\mathbb{R}^3)$. Thus, by Hölder's inequality, $(\vec{u}(s)\cdot\nabla)\vec{v}(s) + (\vec{v}(s)\cdot\nabla)\vec{u}(s) \in L^2(\Omega,\mathbb{R}^3)$ for each $\vec{u},\vec{v} \in \mathcal{F}_T$ and, therefore, $\vec{f}(s) \in \mathcal{H}$ for $s \in (0,T)$, with

$$\begin{aligned}
\sup_{0<s<T} s^{\frac34}\|\vec{f}(s)\|_{\mathcal{H}} &\leq \sup_{0<s<T}\Big\{ s^{\frac34}\Big(\|\vec{u}(s)\|_{L^3_1(\Omega,\mathbb{R}^3)}\|\vec{v}(s)\|_{L^6(\Omega,\mathbb{R}^3)} \\
&\qquad + \|\vec{v}(s)\|_{L^3_1(\Omega,\mathbb{R}^3)}\|\vec{u}(s)\|_{L^6(\Omega,\mathbb{R}^3)}\Big)\Big\} \\
&\leq C\sup_{0<s<T}\Big\{ s^{\frac34}\Big(\|\vec{u}(s)\|_{D(B_\lambda^{\frac34})}\|\vec{v}(s)\|^{1/2}_{D(B_\lambda^{\frac14})}\|\vec{v}(s)\|^{1/2}_{D(B_\lambda^{\frac34})} \\
&\qquad + \|\vec{v}(s)\|_{D(B_\lambda^{\frac34})}\|\vec{u}\|^{1/2}_{D(B_\lambda^{\frac14})}\|\vec{u}(s)\|^{1/2}_{D(B_\lambda^{\frac34})}\Big)\Big\} \\
&\leq C\sup_{0<s<T}\Big\{ s^{\frac34}\Big(\|B_\lambda^{\frac34}\vec{u}(s)\|\|B_\lambda^{\frac14}\vec{v}(s)\|^{1/2}_{\mathcal{H}}\|B_\lambda^{\frac34}\vec{v}(s)\|^{1/2}_{\mathcal{H}} \\
&\qquad + \|B_\lambda^{\frac34}\vec{v}(s)\|_{\mathcal{H}}\|B_\lambda^{\frac14}\vec{u}(s)\|^{1/2}_{\mathcal{H}}\|B_\lambda^{\frac34}\vec{u}(s)\|^{1/2}_{\mathcal{H}}\Big)\Big\} \\
&\leq C\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T}. \qquad (11.13)
\end{aligned}$$

Based on (11.13) and (5.11) we may then estimate

$$\begin{aligned}
\|B_\lambda^{\frac14}\Phi(\vec{u},\vec{v})(t)\|_{\mathcal{H}} &\leq \int_0^t \|B_\lambda^{\frac14}e^{-(t-s)B_\lambda}\|_{\mathscr{L}(\mathcal{H})}\|\vec{f}(s)\|_{\mathcal{H}}\,ds \\
&\leq C\Big(\int_0^t (t-s)^{-\frac14}s^{-\frac34}\,ds\Big)\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\
&\leq C\Big(\int_0^1 (1-\sigma)^{-\frac14}\sigma^{-\frac34}\,d\sigma\Big)\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\
&\leq C\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T}. \qquad (11.14)
\end{aligned}$$

In order to check that the application $[0,T] \ni t \mapsto \Phi(\vec{u},\vec{v})(t) \in D(B_\lambda^{\frac14})$ is continuous, fix an arbitrary $t_o \in [0,T]$ and estimate $\|B_\lambda^{\frac14}\Phi(\vec{u},\vec{v})(t) - B_\lambda^{\frac14}\Phi(\vec{u},\vec{v})(t_o)\|_{\mathcal{H}}$ by distinguishing two scenarios: $0 \leq t \leq t_o$, and $t_o \leq t \leq T$. In the first case, we recall a general identity to the effect that

$$e^{-tB_\lambda}\vec{w} - e^{-t_oB_\lambda}\vec{w} = B_\lambda\left(\int_t^{t_o} e^{-\tau B_\lambda}\vec{w}\,d\tau\right), \qquad \forall\,\vec{w} \in \mathcal{H}. \tag{11.15}$$

Cf. [35, (2.4), pp. 5]. Formula (11.15) allows us to write

$$\begin{aligned}
&B_\lambda^{\frac14}\Phi(\vec{u},\vec{v})(t) - B_\lambda^{\frac14}\Phi(\vec{u},\vec{v})(t_o) \\
&\quad = B_\lambda^{\frac14}\int_0^t B_\lambda\left(\int_t^{t_o} e^{-(\tau-s)B_\lambda}\vec{f}(s)\,d\tau\right)ds + \int_t^{t_o} B_\lambda^{\frac14}e^{-(t-s)B_\lambda}\vec{f}(s)\,ds \\
&\quad =: \mathcal{I}_1 + \mathcal{I}_2. \qquad (11.16)
\end{aligned}$$

Now,

$$\begin{aligned} \|\mathcal{I}_1\|_{\mathcal{H}} &\leq C \sup_{0<s<T}\Big[s^{\frac{3}{4}}\|\vec{f}(s)\|_{\mathcal{H}}\Big]\left[\int_0^t\left(\int_t^{t_o}\frac{d\tau}{(\tau-s)^{5/4}}\right)s^{-\frac{3}{4}}\,ds\right] \\ &\leq C\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T}\int_0^t\Big[(t_o-s)^{-\frac{1}{4}}-(t-s)^{-\frac{1}{4}}\Big]s^{-\frac{3}{4}}\,ds \xrightarrow[t\nearrow t_o]{} 0, \end{aligned} \tag{11.17}$$

and

$$\|\mathcal{I}_2\|_{\mathcal{H}} \leq C\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T}\Big(\int_t^{t_o}(t-s)^{-\frac{1}{4}}s^{-\frac{3}{4}}\,ds\Big) \xrightarrow[t\nearrow t_o]{} 0. \tag{11.18}$$

Thus, altogether, $\|B_\lambda^{\frac{1}{4}}\Phi(\vec{u},\vec{v})(t)-B_\lambda^{\frac{1}{4}}\Phi(\vec{u},\vec{v})(t_o)\|_{\mathcal{H}} \xrightarrow[t\nearrow t_o]{} 0$. In fact, the same is true when $t\searrow t_o$ and this ultimately shows that

$$\Phi(\vec{u},\vec{v})\in\mathscr{C}([0,T];D(B_\lambda^{\frac{1}{4}})) \quad\text{and}\quad \sup_{0<t<T}\|B_\lambda^{\frac{1}{4}}\Phi(\vec{u},\vec{v})(t)\|_{\mathcal{H}}\leq C\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \tag{11.19}$$

for every $\vec{u},\vec{v}\in\mathcal{F}_T$, where $C>0$ is a finite constant, independent of $T>0$.

Going further, we estimate

$$\begin{aligned} \|B_\lambda^{\frac{3}{4}}\Phi(\vec{u},\vec{v})(t)\|_{\mathcal{H}} &\leq \int_0^t\|B_\lambda^{\frac{3}{4}}e^{-(t-s)B_\lambda}\|_{\mathscr{L}(\mathcal{H})}\|\vec{f}(s)\|_{\mathcal{H}}\,ds \\ &\leq C\left(\int_0^t(t-s)^{-\frac{3}{4}}s^{-\frac{3}{4}}\,ds\right)\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\ &\leq C\,t^{-\frac{1}{2}}\Big(\int_0^1(1-\sigma)^{-\frac{3}{4}}\sigma^{-\frac{3}{4}}\,d\sigma\Big)\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\ &\leq C\,t^{-\frac{1}{2}}\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T}. \end{aligned} \tag{11.20}$$

The continuity of the map $(0,T]\ni t\mapsto B_\lambda^{\frac{3}{4}}\Phi(\vec{u},\vec{v})(t)\in\mathcal{H}$ can then be established as before. In order to estimate the derivative in time of $\Phi(\vec{u},\vec{v})(t)$, we first note that for each $s\in(0,T)$

$$\vec{f}'(s)=(-\tfrac{1}{2}\mathbb{P})((\vec{u}'(s)\cdot\nabla)\vec{v}(s)+(\vec{u}(s)\cdot\nabla)\vec{v}'(s)+(\vec{v}'(s)\cdot\nabla)\vec{u}(s)+(\vec{v}(s)\cdot\nabla)\vec{u}'(s)). \tag{11.21}$$

In particular, much as in (11.13),

$$\sup_{0<s<T}s^{\frac{7}{4}}\|\vec{f}'(s)\|_{\mathcal{H}}\leq C\,\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T}. \tag{11.22}$$

where $C>0$ is independent of T. After this preamble we write

$$\Phi(\vec{u},\vec{v})(t)=\int_0^{\frac{t}{2}}e^{-sB_\lambda}\vec{f}(t-s)\,ds+\int_0^{\frac{t}{2}}e^{-(t-s)B_\lambda}\vec{f}(s)\,ds,\quad t\in]0,T[, \tag{11.23}$$

and, therefore,

$$\Phi(\vec{u},\vec{v})'(t)=e^{-\frac{t}{2}B_\lambda}\vec{f}(\tfrac{t}{2})+\int_0^{\frac{t}{2}}e^{-sB_\lambda}\vec{f}'(t-s)\,ds+\int_0^{\frac{t}{2}}-B_\lambda e^{-(t-s)B_\lambda}\vec{f}(s)\,ds. \tag{11.24}$$

In concert with (11.13) and (11.22), this allows us to estimate

$$
\begin{aligned}
\|\Phi(\vec{u},\vec{v})'(t)\|_{\mathcal{H}} &\leq C\,\|\vec{f}(\tfrac{t}{2})\|_{\mathcal{H}} + C\int_0^{\frac{t}{2}} \|-B_\lambda e^{-(t-s)B_\lambda}\|_{\mathscr{L}(\mathcal{H})}\|\vec{f}(s)\|_{\mathcal{H}}\,ds \\
&\quad + C\int_0^{\frac{t}{2}} \|e^{-sB_\lambda}\|_{\mathscr{L}(\mathcal{H})}\|\vec{f}'(t-s)\|_{\mathcal{H}}\,ds \\
&\leq C\,t^{-\frac{3}{4}}\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} + C\int_0^{\frac{t}{2}}(t-s)^{-1}s^{-\frac{3}{4}}\,ds\,\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\
&\quad + C\int_0^{\frac{t}{2}}(t-s)^{-\frac{7}{4}}\,ds\,\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\
&\leq C\,t^{-\frac{3}{4}}\Big(1+\int_0^{\frac{1}{2}}(1-\sigma)^{-\frac{7}{4}}\,d\sigma + \int_0^{\frac{1}{2}}(1-\sigma)^{-1}\sigma^{-\frac{3}{4}}\,d\sigma\Big)\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\
&\leq C\,t^{-\frac{3}{4}}\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T},
\end{aligned}
\tag{11.25}
$$

where $C>0$ is independent of T. Furthermore, by reasoning as before, one can show that the application $(0,T]\ni t\mapsto \Phi(\vec{u},\vec{v})'(t)\in D(B_\lambda^{\frac{3}{4}})$ is continuous.

Finally,

$$
\begin{aligned}
\|B_\lambda^{\frac{3}{4}}\Phi(\vec{u},\vec{v})'(t)\|_{\mathcal{H}} &\leq C\|B_\lambda^{\frac{3}{4}}e^{-\frac{t}{2}B_\lambda}\|_{\mathscr{L}(\mathcal{H})}\|\vec{f}(\tfrac{t}{2})\|_{\mathcal{H}} + C\int_0^{\frac{t}{2}} \|-B_\lambda^{\frac{7}{4}}e^{-(t-s)B_\lambda}\|_{\mathscr{L}(\mathcal{H})}\|\vec{f}(s)\|_{\mathcal{H}}\,ds \\
&\quad + C\int_0^{\frac{t}{2}} \|B_\lambda^{\frac{3}{4}}e^{-sB_\lambda}\|_{\mathscr{L}(\mathcal{H})}\|\vec{f}'(t-s)\|_{\mathcal{H}}\,ds \\
&\leq C\,t^{-\frac{3}{2}}\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} + C\int_0^{\frac{t}{2}}(t-s)^{-\frac{7}{4}}s^{-\frac{3}{4}}\,ds\,\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\
&\quad + C\int_0^{\frac{t}{2}}(t-s)^{-\frac{7}{4}}s^{-\frac{3}{4}}\,ds\,\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\
&\leq C\,t^{-\frac{3}{2}}\Big(1+\int_0^{\frac{1}{2}}(1-\sigma)^{-\frac{7}{4}}\sigma^{-\frac{3}{4}}\,d\sigma + \int_0^{\frac{1}{2}}(1-\sigma)^{-\frac{7}{4}}\sigma^{-\frac{3}{4}}\,d\sigma\Big)\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T} \\
&\leq C\,t^{-\frac{3}{2}}\|\vec{u}\|_{\mathcal{F}_T}\|\vec{v}\|_{\mathcal{F}_T},
\end{aligned}
\tag{11.26}
$$

where, once again, the constant C does not dependent of T.

The above analysis ensures that $\Phi(\vec{u},\vec{v})\in\mathcal{F}_T$ whenever $\vec{u},\vec{v}\in\mathcal{F}_T$. Moreover, from (11.19), (11.20), (11.25) and (11.26), there exists a constant $\kappa>0$ independent of $T>0$ such that (11.11) holds. □

We are now ready to discuss the existence of *mild solutions* for the Navier-Stokes system.

Theorem 11.3. *Given $\vec{u}_0\in D(A^{\frac{1}{4}})$ and $T>0$, the equation*

$$
u(t)=e^{-tB_\lambda}u_0+\Phi(\vec{u},\vec{u})(t),\qquad 0<t<T,
\tag{11.27}
$$

has a unique solution $\vec{u}\in\mathcal{F}_T$, if either $\|\vec{u}_0\|_{D(A^{\frac{1}{4}})}$ or T are sufficiently small.

Proof. Let $T>0$ be given and consider the bilinear, continuous mapping $\Phi:\mathcal{F}_T\times\mathcal{F}_T\to\mathcal{F}_T$ defined as in (11.9). As in [11], a solution of (11.27) will be found implementing

Picard's fixed point theorem. That is, consider the sequence in $\{\vec{v}_j\}_j$ of functions in $\mathcal{F}_T$ defined by $\vec{v}_0 := S\vec{u}_0$ and

$$\vec{v}_{j+1} := \vec{v}_0 + \Phi(\vec{v}_j, \vec{v}_j), \qquad j \in \mathbb{N}. \tag{11.28}$$

As is well-known (cf., e.g., [26, Lemma 20, p. 157]), this sequence converges to the unique solution $\vec{u} \in \mathcal{F}_T$ of (11.27) provided

$$\|\vec{v}_0\|_{\mathcal{F}_T} < \frac{1}{4\kappa}, \tag{11.29}$$

where κ is the constant appearing in (11.11). In turn, since $\|\vec{v}_0\|_{\mathcal{F}_T} \leq C\|B_\lambda^{\frac{1}{4}}\vec{u}_0\|_{\mathcal{H}}$, the estimate (11.29) is satisfied granted that $\|\vec{u}_0\|_{D(B_\lambda^{\frac{1}{4}})}$ is small enough.

To finish the proof, it suffices to show that, irrespective of the size of $\|\vec{u}_0\|_{D(B_\lambda^{\frac{1}{4}})}$, matters can be arranged so that (11.29) holds by taking T small enough (relative to $\|\vec{u}_0\|_{D(B_\lambda^{\frac{1}{4}})}$). To see this, we shall make use of the fact that for each $\varepsilon > 0$ there exists $\vec{u}_{0,\varepsilon} \in D(B_\lambda)$ such that $\|B_\lambda^{\frac{1}{4}}(\vec{u}_0 - \vec{u}_{0,\varepsilon})\|_{\mathcal{H}} \leq \varepsilon$. If we now consider $\vec{v}_{0,\varepsilon}(t) := S\vec{u}_{0,\varepsilon}$ for $0 < t < T$, then

$$\|\vec{v}_0 - \vec{v}_{0,\varepsilon}\|_{\mathcal{F}_T} \leq C\|B_\lambda^{\frac{1}{4}}(u_0 - u_{0,\varepsilon})\|_{\mathcal{H}} \leq C\varepsilon, \tag{11.30}$$

by (11.4) and, for each fixed ε,

$$\|\vec{v}_{0,\varepsilon}\|_{\mathcal{F}_T} \leq C\,T^{\frac{3}{4}}\|B_\lambda \vec{u}_{0,\varepsilon}\|_{\mathcal{H}} \xrightarrow[T\to 0^+]{} 0. \tag{11.31}$$

By first choosing $\varepsilon > 0$ small enough, we can therefore find $T > 0$ such that (11.29) is valid. This concludes the proof of the theorem. $\square$

Remark 11.4. A somewhat smaller space for which the analogues of (11.4) and (11.10) hold is as follows

$$\mathcal{F}_T^0 := \{\vec{u} \in \mathcal{F}_T : \lim_{\tau\to 0^+} \|\vec{u}\|_{\mathcal{F}_\tau} = 0\}. \tag{11.32}$$

11.2. **Regularity.** Here, we shall prove that the solution $\vec{u} \in \mathcal{F}_T$ of the fixed point problem (11.27) is actually a solution of the Navier-Stokes system (1.13) in the suitable sense, made precise in the theorem below.

Theorem 11.5. *Any solution $\vec{u} \in \mathcal{F}_T$ of the problem* (11.27) *satisfies $\vec{u}(0) = \vec{u}_0$ in Ω and, in addition, has the following properties. For every $t \in [0,T]$, the field $\vec{u}(t,\cdot)$ is divergence free in Ω and there exists $\pi \in L^p(0,T;L^2(\Omega))$ such that $-\Delta_x u + \nabla_x \pi \in L^p(0,T;L^2(\Omega)^3)$, $(\vec{u},\pi)$ has vanishing conormal derivative (cf. § 3) on $\partial\Omega$, and for which the first equation in* (1.13) *is satisfied everywhere in the time variable $t \in (0,T]$ and almost everywhere in the space variable $x \in \Omega$. Furthermore,*

$$\vec{u} \in L_1^p(0,T;\mathcal{H}) \cap L^p(0,T;D(B_\lambda)), \qquad 1 < p < \tfrac{4}{3}, \tag{11.33}$$

and matters can be arranged so that

$$\lim_{\tau\to 0^+} \|\vec{u}\|_{\mathcal{F}_\tau} = 0. \tag{11.34}$$

Proof. Assume that $\vec{u} \in \mathcal{F}_T$ solves (11.27) and introduce

$$\vec{f}(s) := -\mathbb{P}\big[(\vec{u}(s)\cdot\nabla_x)\vec{u}(s)\big], \qquad s \in [0,T]. \tag{11.35}$$

From (11.13) we may conclude that $\vec{f}\in L^p(0,T;\mathcal{H})$ whenever $1<p<\frac{4}{3}$ and, from (11.27), that $\vec{u}=e^{-\cdot B_\lambda}\vec{u}_0+e^{-\cdot B_\lambda}*\vec{f}$. Now, the maximal regularity property for the Neumann-Stokes operator in $\mathcal{H}$ (since $-B_\lambda$ generates an analytic semigroup in the Hilbert space $\mathcal{H}$; cf. [6]) and the fact that $\vec{u}_0\in D(B_\lambda^{\frac14})$ entail $B_\lambda\vec{u}\in L^p(0,T;\mathcal{H})$ and that $\vec{u}$ solves

$$\vec{u}'(t)+(B_\lambda\vec{u})(t)=\vec{f}(t)\ \text{ for a.e. } t\in(0,T), \text{ and }\ \vec{u}(0)=\vec{u}_0. \tag{11.36}$$

It follows from the definition of the Neumann-Stokes operator B_λ (cf. (6.2)-(6.3)) that there exists $q_1\in L^p(0,T;L^2(\Omega))$ such that

$$L^p(0,T;L^2(\Omega)^3)\ni B_\lambda\vec{u}=-\Delta_x\vec{u}+\nabla_x q_1 \tag{11.37}$$

$$L^p(0,T;L^2_{-1/2}(\partial\Omega)^3)\ni\partial_\nu^\lambda(u,q_1)=0. \tag{11.38}$$

Moreover, by the definition of $\mathbb{P}$ ((7.22) for $p=2$ and $s=0$), there exists $q_2\in L^p(0,T;L^2_{1,z}(\Omega)^3)$ such that

$$\mathbb{P}[(\vec{u}\cdot\nabla_x)\vec{u}]=(\vec{u}\cdot\nabla_x)\vec{u}-\nabla q_2. \tag{11.39}$$

Since $(\vec{u}\cdot\nabla_x)\vec{u}\in\mathscr{C}((0,T];L^2(\Omega)^3)$, we also have

$$q_2\in\mathscr{C}((0,T];L^2_{1,z}(\Omega)). \tag{11.40}$$

Combining (11.37)-(11.39) with (11.36) yields (1.13) with $\pi=q_1-q_2$. Moreover, since $\vec{u}'\in\mathscr{C}((0,T];\mathcal{H})$ and $\vec{f}\in\mathscr{C}((0,T];\mathcal{H})$, we may finally conclude from (11.37) and (11.40) that $-\Delta_x u+\nabla_x\pi\in\mathscr{C}((0,T];L^2(\Omega)^3)$. Thus, the Navier-Stokes system (1.13)) holds as mentioned, whereas (11.34) is a consequence of the remark made at the end of §11.1. □

11.3. **Uniqueness.** We have already proved that there exists a local mild solution to the Navier-Stokes system which is unique in the space $\mathcal{F}_T$. Following [33], here we shall prove that, in fact, uniqueness holds in the larger space $\mathscr{C}([0,T];D(B_\lambda^{\frac14}))$.

Prior to formally stating this as a theorem, need to make sense of the non-linearity $\Phi(\vec{u},\vec{u})$ for fields $\vec{u}\in\mathscr{C}([0,T];D(B_\lambda^{\frac14}))$. To this end, for $\vec{u},\vec{v}\in\mathscr{C}([0,T];D(B_\lambda^{\frac14}))$ consider

$$\vec{f}(s):=\Big(-\tfrac12\widehat{\mathbb{P}}\nabla_x\cdot\Big)\Big(\vec{u}(s)\otimes\vec{v}(s)+\vec{v}(s)\otimes\vec{u}(s)\Big),\quad s\in(0,T), \tag{11.41}$$

where, generally speaking, $\vec{a}\otimes\vec{b}$ denotes the matrix $(a_ib_j)_{1\le i,j\le 3}$ for any $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)\in\mathbb{R}^3$. In this connection, let us also note that if $\vec{a}$ and $\vec{b}$ are smooth vector fields then

$$\nabla_x\cdot(\vec{a}\otimes\vec{b})=(\vec{a}\cdot\nabla_x)\vec{b}+(\operatorname{div}\vec{a})\vec{b}. \tag{11.42}$$

This elementary identity allows us to extend the bilinear form Φ, originally defined on $\mathcal{F}_T\times\mathcal{F}_T$, to the larger space $\mathscr{C}([0,T];D(B_\lambda^{\frac14}))\times\mathscr{C}([0,T];D(B_\lambda^{\frac14}))$ in the following sense. First, if $\vec{u},\vec{v}\in\mathscr{C}([0,T];D(B_\lambda^{\frac14}))$ are arbitrary then both $\vec{u}\otimes\vec{v}$ and $\vec{v}\otimes\vec{u}$ belong to $\mathscr{C}([0,T];L^{\frac32}(\Omega)^{3\times3})$, since $D(B_\lambda^{\frac14})\subset L^3(\Omega)^3$. In particular,

$$\nabla_x\cdot(\vec{u}\otimes\vec{v}+\vec{v}\otimes\vec{u})\in\mathscr{C}([0,T];L^{\frac32}_{-1}(\Omega)^3). \tag{11.43}$$

We now digress momentarily in order to establish a useful auxiliary result.

Lemma 11.6. *The operator $\widehat{\mathbb{P}}$, introduced in Lemma 10.1, has the property that*

$$B_\lambda^{-\frac34}\widehat{\mathbb{P}}:L^{\frac32}_{-1}(\Omega)^3\longrightarrow\mathcal{H} \tag{11.44}$$

in a bounded fashion.

Proof. Using (3.13) and (3.20), we know from (10.1) that $\widehat{\mathbb{P}}$ maps $L^{\frac{3}{2}}_{-1}(\Omega)^3$ boundedly into the space $(V^{1,3}(\Omega))^*$ which, in turn, embeds continuously into $D(B_\lambda^{\frac{3}{4}})^*$ by (10.38). Since B_λ is self-adjoint, we also have $B_\lambda^{-\frac{3}{4}}\big[D(B_\lambda^{\frac{3}{4}})^*\big]=\mathcal{H}$, and (11.44) follows. □

Returning to the mainstream discussion, we note that $B_\lambda^{-\frac{3}{4}}f\in\mathscr{C}([0,T];\mathcal{H})$, by (11.43) and Lemma 11.6. Therefore, writing

$$\Phi(\vec{u},\vec{v})(t)=\int_0^t B_\lambda^{\frac{3}{4}}e^{-(t-s)B_\lambda}B_\lambda^{-\frac{3}{4}}\vec{f}(s)\,ds,\qquad t\in[0,T], \tag{11.45}$$

it follows that

$$\Phi:\mathscr{C}([0,T];D(B_\lambda^{\frac{1}{4}}))\times\mathscr{C}([0,T];D(B_\lambda^{\frac{1}{4}}))\longrightarrow\mathscr{C}([0,T],\mathcal{H}) \tag{11.46}$$

in a bilinear, bounded fashion. Another useful property of this map is as follows.

Proposition 11.7. *For each $p\in(1,\infty)$ the mapping* (11.46) *further extends to a bounded bilinear application*

$$\Phi:L^p(0,T;D(B_\lambda^{\frac{1}{4}}))\times L^\infty(0,T;D(B_\lambda^{\frac{1}{4}}))\longrightarrow L^p(0,T;D(B_\lambda^{\frac{1}{4}})). \tag{11.47}$$

Furthermore, the norm of (11.47) *is bounded by a constant which depends exclusively on p.*

Proof. For $\vec{u}\in L^p(0,T;D(B_\lambda^{\frac{1}{4}}))$ and $\vec{v}\in L^\infty(0,T;D(B_\lambda^{\frac{1}{4}}))$, the function f defined in (11.41) satisfies the estimate

$$\|B_\lambda^{-\frac{3}{4}}\vec{f}\|_{L^p(0,T;\mathcal{H})}\le C_p\|B_\lambda^{\frac{1}{4}}\vec{u}\|_{L^p(0,T;\mathcal{H})}\|B_\lambda^{\frac{1}{4}}\vec{v}\|_{L^\infty(0,T;\mathcal{H})} \tag{11.48}$$

for a finite constant $C_p>0$. Then, thanks to the maximal regularity property for B_λ, we have

$$B_\lambda^{\frac{1}{4}}\Phi(\vec{u},\vec{v})=B_\lambda(e^{-\cdot B_\lambda}*B_\lambda^{-\frac{3}{4}}\vec{f})\in L^p(0,T;\mathcal{H}) \tag{11.49}$$

and

$$\|B_\lambda^{\frac{1}{4}}\Phi(\vec{u},\vec{v})\|_{L^p(0,T;\mathcal{H})}\le C_p\|B_\lambda^{\frac{1}{4}}\vec{u}\|_{L^p(0,T;\mathcal{H})}\|B_\lambda^{\frac{1}{4}}\vec{v}\|_{L^\infty(0,T;\mathcal{H})}, \tag{11.50}$$

as desired. □

We are now in a position to discuss the uniqueness of mild solutions for the Navier-Stokes system, which is the main result of this subsection. To state it formally, for a measurable set $E\subset\mathbb{R}$ and a Banach space $\mathscr{X}$, we set $\mathscr{C}_b(E;\mathscr{X}):=\mathscr{C}(E,;\mathscr{X})\cap L^\infty(E;\mathscr{X})$.

Theorem 11.8. *For each $\vec{u}_0\in D(B_\lambda^{\frac{1}{4}})$, there is at most one field $\vec{u}\in\mathscr{C}_b([0,T);D(B_\lambda^{\frac{1}{4}}))$ which satisfies* (11.27).

Proof. Assume that for some $\vec{u}_0\in D(B_\lambda^{\frac{1}{4}})$ there exist two vector fields $\vec{u}_1$, $\vec{u}_2$ which belong to $\mathscr{C}_b([0,T);D(B_\lambda^{\frac{1}{4}}))$ and which solve (11.27). Then $\vec{w}:=\vec{u}_1-\vec{u}_2$ also belongs to $\mathscr{C}_b([0,T);D(B_\lambda^{\frac{1}{4}}))$ and, in addition, satisfies

$$\vec{w}=\Phi(\vec{u}_1,\vec{u}_1)-\Phi(\vec{u}_2,\vec{u}_2)=\Phi(\vec{w},\vec{u}_1+\vec{u}_2)=\Phi(\vec{w},\vec{u}_1+\vec{u}_2-2S\vec{u}_0)+2\Phi(\vec{w},S\vec{u}_0), \tag{11.51}$$

where S is the Stokes semigroup (cf. (11.3)).

The traditional strategy (cf., e.g., [33] and the references therein) is to prove that, for a fixed $p \in (1,\infty)$, there exists $\tau \in (0,T]$ such that

$$\|\vec{w}\|_{L^p(0,\tau;D(B_\lambda^{\frac14}))} \leq \frac{\|\vec{w}\|_{L^p(0,\tau;D(B_\lambda^{\frac14}))}}{2}. \tag{11.52}$$

Granted this estimate, we may conclude that $\vec{w}$ vanishes on $[0,\tau)$ which, in turn, proves that $\{\tau \in (0,T] : \vec{w}(t) = 0 \text{ for } 0 \leq t < \tau\}$ is nonempty. Let us denote its supremum by τ_{max}. If $\tau_{max} < T$, the continuity of $\vec{w}$ entails $\vec{w}(\tau_{max}) = 0$. In this scenario, the above scheme can be reiterated, taking τ_{max} as the initial time, and we eventually conclude that there exists some $\delta > 0$ such that $\vec{w} = 0$ on $[0, \tau_{max} + \delta)$. This contradicts the maximality of τ_{max} and proves that $\tau_{max} = T$. Thus $\vec{w} = 0$ on $[0,T]$, as wanted.

There remains to establish (11.52). For starters, we note that for any $p \in (1,\infty)$, Proposition 11.7 gives

$$\begin{aligned} &\|\Phi(\vec{w}, \vec{u}_1 + \vec{u}_2 - 2S\vec{u}_0)\|_{L^p(0,\tau;D(B_\lambda^{\frac14}))} \\ &\quad \leq C_p \|\vec{w}\|_{L^p(0,\tau;D(B_\lambda^{\frac14}))} \Big(\|\vec{u}_1 - S\vec{u}_0\|_{L^\infty(0,\tau;D(B_\lambda^{\frac14}))} + \|\vec{u}_2 - S\vec{u}_0\|_{L^\infty(0,\tau;D(B_\lambda^{\frac14}))} \Big). \end{aligned} \tag{11.53}$$

Since

$$\|\vec{u}_j - S\vec{u}_0\|_{L^\infty(0,\tau;D(A^{\frac14}))} \xrightarrow[\tau\to 0^+]{} 0, \qquad j = 1,2, \tag{11.54}$$

it follows that (11.53) is useful for the purpose of establishing (11.52).

There remains to handle the term $2\Phi(\vec{w}, S\vec{u}_0)$. To this end, for an arbitrary $\varepsilon > 0$, to be specified later, pick $\vec{u}_{0,\varepsilon} \in D(B_\lambda)$ such that $\|\vec{u}_0 - \vec{u}_{0,\varepsilon}\|_{D(B_\lambda^{\frac14})} < \varepsilon$ and then write

$$\begin{aligned} &\|\Phi(\vec{w}, S\vec{u}_0)\|_{L^p(0,\tau;D(B_\lambda^{\frac14}))} \\ &\quad \leq C_p \|\vec{w}\|_{L^p(0,\tau;D(B_\lambda^{\frac14}))} \Big(\|S(\vec{u}_0 - \vec{u}_{0,\varepsilon})\|_{L^\infty(0,\tau;D(B_\lambda^{\frac14}))} + \|S\vec{u}_{0,\varepsilon}\|_{L^\infty(0,\tau;D(B_\lambda^{\frac14}))} \Big). \end{aligned} \tag{11.55}$$

Next,

$$\|S(\vec{u}_0 - \vec{u}_{0,\varepsilon})\|_{L^\infty(0,\tau;D(B_\lambda^{\frac14}))} \leq \|\vec{u}_0 - \vec{u}_{0,\varepsilon}\|_{D(B_\lambda^{\frac14})} < \varepsilon \tag{11.56}$$

Finally, much as with (11.31),

$$\|S\vec{u}_{0,\varepsilon}\|_{L^\infty(0,\tau;D(B_\lambda^{\frac14}))} \leq C\,\tau^{\frac34} \|B_\lambda \vec{u}_{0,\varepsilon}\|_{\mathcal{H}} \xrightarrow[\tau\to 0^+]{} 0. \tag{11.57}$$

In summary, by first choosing $\varepsilon > 0$ small enough (relative to the constant C_p in (11.55)) it is then possible to ensure that (11.52) holds provided $\tau > 0$ is sufficiently small. This justifies (11.52) and concludes the proof of the theorem. □

References

[1] W. Arendt, *Semigroups and evolution equations: functional calculus, regularity and kernel estimates*, pp. 1–85 in *Evolutionary Equations*, Vol. I, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[2] R. Brown and Z. Shen, *Estimates for the Stokes operator in Lipschitz domains*, Indiana Univ. J., 44 (1995), 1183–1206.

[3] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Volume 2, Functional and Variational Methods*, Springer, Berlin, 2000.

[4] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Volume 5, Evolution Problems I*, Springer, Berlin, 2000.

[5] G. de Rham, *Differentiable Manifolds. Forms, Currents, Harmonic Forms*, Springer-Verlag, Berlin, 1984.

[6] L. de Simon, *Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine*, *Rend. Sem. Mat. Univ. Padova*, 34 (1964), 205–223.

[7] D.E. Edmunds and W.D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1989.

[8] E.B. Fabes, C.E. Kenig and G.C. Verchota *The Dirichlet problem for the Stokes system on Lipschitz domains*, Duke Math. J. 57 (1988), no. 3, 769–793.

[9] E. Fabes, O. Mendez and M. Mitrea *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*, J. Funct. Anal., 159 (1998), 323–368.

[10] W.G. Faris, *Self-Adjoint Operators*, Lecture Notes in Mathematics, Vol. 433, Springer, Berlin, 1975.

[11] H. Fujita and T. Kato, *On the Navier-Stokes initial value problem, I*, Arch. Rational Mech. Anal., 16 (1964), 269–315.

[12] H. Fujita and H. Morimoto, *On fractional powers of the Stokes operator*, Proc. Japan Acad., 46 (1970), 1141–1143.

[13] Y. Giga and T. Miyakawa, *Solutions in L^r of the Navier-Stokes initial value problem*, Arch. Rational Mech. Anal., 89 (1985), no. 3, 267–281.

[14] G. Grubb, *Initial value problems for the Navier-Stokes equations with Neumann conditions*, pp. 262–283 in "The Navier-Stokes Equations II—Theory and Numerical Methods" (Oberwolfach, 1991), Lecture Notes in Math., 1530, Springer, Berlin, 1992.

[15] G. Grubb and V.A. Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand., 69 (1991), no. 2, 217–290.

[16] D. Jerison, C. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains, *J. Funct. Anal.*, 130:161–219, 1995.

[17] D.D. Joseph, *Stability of Fluid Motions*, Vol. II, Springer Tracts in Natural Philosophy, 28, Springer-Verlag, Berlin-New York, 1976.

[18] T. Kato, *Fractional powers of dissipative operators. I*, J. Math. Soc. Japan, 13 (1961), 246–274.

[19] T. Kato, *Fractional powers of dissipative operators. II*, J. Math. Soc. Japan, 14 (1962), 242–248.

[20] T. Kato, *Perturbation Theory for Linear Operators*, corr. printing of the 2-nd ed., Springer, Berlin, 1980.

[21] T. Kato, M. Mitrea G. Ponce and M. Taylor, *Extension and representation of divergence-free vector fields on bounded domains*, Mathematical Research Letters, 7 (2000) no. 5-6, 643–650.

[22] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, Science Publishers, New York-London-Paris, 1969.

[23] J.-L. Lions, *Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs*, J. Math. Soc. Japan, Vol. 14, (1962), 233–241.

[24] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. 1, Travaux et Recherches Mathématiques, No. 17 Dunod, Paris 1968.

[25] S. Mayboroda, M. Mitrea. *Sharp estimates for Green potentials on non-smooth domains*, Mathematical Research Letters, 11 (2004), 481–492.

[26] Y. Meyer. Wavelets, paraproducts, and Navier-Stokes equations. pp. 105–212 in *Current Developments in Mathematics*, Int. Press, Boston, MA, 1997.

[27] D. Mitrea, M. Mitrea and S. Monniaux, *The Poisson problem for the exterior derivative operator with Dirichlet boundary condition on nonsmooth domains*, Commun. Pure Appl. Anal., 7 (2008), no. 6, 1295–1333.

[28] M. Mitrea, *Sharp Hodge decompositions, Maxwell's equations, and vector Poisson problems on nonsmooth, three-dimensional Riemannian manifolds*, Duke Math. J., 125 (2004), no. 3, 1–81.

[29] M. Mitrea and S. Monniaux, *The regularity of the Stokes operator and the Fujita-Kato approach to the Navier-Stokes initial value problem in Lipschitz domains*, J. Funct. Anal., 254 (2008), no. 6, 1522–1574.

[30] M. Mitrea and S. Monniaux, *The nonlinear Hodge-Navier-Stokes equations in Lipschitz domains*, Differential Integral Equations 22 (2009), no. 3-4, 339–356.

[31] M. Mitrea, S. Monniaux and M. Wright, in preparation, (2008).

[32] M. Mitrea and M. Wright, *Boundary Value Problems for the Stokes System in Arbitrary Lipschitz Domains*, to appear in Astérisque, (2010).

[33] S. Monniaux, *Uniqueness of mild solutions of the NavierStokes equation and maximal L^p-regularity*, C. R. Acad. Sci. Paris Sr. I Math. 328 (1999) 663668.

[34] E.M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Mathematical Society Monographs Series, Vol. 31, Princeton University Press, Princeton, NJ, 2005.

[35] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, Vol. 44, Springer-Verlag, New York, 1983.

[36] H. Power and L.C. Wrobel, *Boundary Integral Methods in Fluid Mechanics*, Computational Mechanics Publications, Southampton, 1995.

[37] V. Rychkov, *On restrictions and extensions of the Besov and Triebel-Lizorkin spaces with respect to Lipschitz domains*, J. London Math. Soc., (2) 60 (1999), no. 1, 237–257.

[38] V.A. Solonnikov, *Solvability of the problem of the motion of a viscous incompressible fluid that is bounded by a free surface*, Izv. Akad. Nauk SSSR Ser. Mat., 41 (1977), no. 6, 1388–1424.

[39] M.E. Taylor, *Partial Differential Equations*, Springer-Verlag, 1996.

[40] M.E. Taylor, *Incompressible fluid flows on rough domains*, pp. 320–334 in "Semigroups of Operators: Theory and Applications", Progr. Nonlinear Differential Equations Appl., Vol. 42, Birkhäuser, Basel, 2000.

[41] R. Temam, *Navier-Stokes Equations. Theory and Numerical Analysis*, North-Holland Publishing Co., Amsterdam, 1984.

[42] H. Triebel, *Theory of function spaces*, Monographs in Mathematics, Vol. 78, Birkhäuser Verlag, Basel, 1983.

[43] W. von Wahl, *The Equations of Navier-Stokes and Abstract Parabolic Equations*, Aspects of Mathematics, E8. Friedr. Vieweg & Sohn, Braunschweig, 1985.

Department of Mathematics - University of Missouri - Columbia - 202 Mathematical Sciences Building - Columbia, MO 65211, USA

E-mail address: mitream@missouri.edu

LATP - UMR 6632 - Faculté des Sciences et Techniques - Université Paul Cézanne - Avenue Escadrille Normandie Niémen - 13397 Marseille Cédex 20 - France

E-mail address: sylvie.monniaux@univ-cezanne.fr

Department of Mathematics - Missouri State University - 901 South National Avenue - Springfield, MO 65897, USA

E-mail address: MWright@MissouriState.edu